Symplectic areas, quantization, and dynamics in electromagnetic fields

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Abstract

A gauge invariant quantization in a closed integral form is developed over a linear phase space endowed with an inhomogeneous Faraday electromagnetic tensor. An analog of the Groenewold product formula (corresponding to Weyl ordering) is obtained via a membrane magnetic area, and extended to the product of N symbols. The problem of ordering in quantization is related to different configurations of membranes: a choice of configuration determines a phase factor that fixes the ordering and controls a symplectic groupoid structure on the secondary phase space. A gauge invariant solution of the quantum evolution problem for a charged particle in an electromagnetic field is represented in an exact continual form and in the semiclassical approximation via the area of dynamical membranes.

1 Introduction and Overview

The works by Berezin [1], Berry [2, 3], and Marinov [4] have introduced into mathematical physics elegant formulas representing three primary quantum objects (the Weyl noncommutative product, semiclassical eigenfunctions, and the evolution wave functions) in terms of symplectic area of simple two-dimensional surfaces (membranes) whose boundary consists of line segments and pieces of Hamiltonian trajectories in phase space. The area of these membranes is determined with respect to the canonical 2-form

$$\omega_0 = \frac{1}{2} J_{jk} dx^k \wedge dx^j, \qquad x = (q, p) \in \mathbb{R}^{2n}, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Attempts to generalize some of these formulas to phase spaces with a generic symplectic form have been undertaken. For symmetric spaces (see [5]) the opportunity to represent the quantum product via the area of triangle membranes was mentioned by Berezin, and it was actually proved in [6], that the semiclassical phase of the product-generating kernel is given by such an area in this case. Over Kählerian manifolds formulas for the Wick product and for the solutions of stationary or evolution problems via the area of some membranes in the complexification of phase space were obtained in [7, 8, 9] (see also [10]). However, formulas which use only the usual symplectic area are still unavailable for the general case.

In the present paper we analyze three problems related to this topic.

A first and trivial remark, which one can make regarding generalization of Berezin–Berry–Marinov geometrical picture, is that the specific matrix J in the definition of the symplectic form ω_0 can be replaced by an arbitrary skew-symmetric matrix without any changes in the geometrical picture. In particular, one can take J to be the matrix $\begin{bmatrix} F & I \\ -I & 0 \end{bmatrix}$, where the constant block F represents a homogenous electromagnetic field. But the next and more interesting generalization to consider is that of inhomogeneous (not constant) tensors F.

The second natural question is about ordering. All treatments of Berezin, Berry and Marinov picture were made for one specific ordering choice: for the Weyl symmetrization of the noncommutative coordinates (operators). What happens with other possible orderings?

The third question which our paper addresses is the application of such Weyl and non–Weyl symbolic calculus, in the presence of an inhomogeneous field, to solving the Cauchy problem and developing a semiclassical representation.

We begin with the first question and consider the linear phase space $\mathbb{R}^{2n}$ with the following inhomogeneous symplectic form

$$\omega_F = \omega_0 + F, \qquad F = \frac{1}{2} F_{jk}(q)\, dq^k \wedge dq^j. \tag{1.1}$$

In cases $n = 3, n = 4$ the form (1.1) describes the structure of the phase space for charged particles in an electromagnetic field [11, 12]; the additional summand F is the Faraday 2–form multiplied by the charge coupling constant e/c. For simplicity of notation we include this constant in F_{jk}; also note that the order of indices j, k in (1.1) is opposite to that used in some textbooks [13].

In detail one has the following. In the 3-dimensional case with $q = (q^1, q^2, q^3)$ then

$$F_{jk} = \frac{e}{c}\epsilon_{kjl} B^l \qquad (j, k = 1, 2, 3),$$

$$F = \frac{e}{c}\Big(B^1(q)\, dq^2 \wedge dq^3 + B^2(q)\, dq^3 \wedge dq^1 + B^3(q)\, dq^1 \wedge dq^2\Big). \tag{1.2a}$$

Here B is the magnetic field, and $dF = 0$ is equivalent to $\mathrm{div} B = 0$. In the 4-dimensional case with $q = (q^0, q^1, q^2, q^3)$, $q^0 \equiv ct$ one has

$$F_{jk} = \frac{e}{c}\epsilon_{kjl} B^l, \qquad (j, k = 1, 2, 3), \qquad F_{0j} = \frac{e}{c} E_j \qquad (j = 1, 2, 3),$$

$$F = \frac{e}{c}\Big(B^1(t, q)\, dq^2 \wedge dq^3 + B^2(t, q)\, dq^3 \wedge dq^1 + B^3(t, q)\, dq^1 \wedge dq^2\Big) + eE_j(t, q) dq^j \wedge dt. \tag{1.2b}$$

The electric field is denoted by E, and $dF = 0$ is equivalent to the pair of Maxwell equations $c^{-1}\partial B/\partial t + \mathrm{curl} E = 0$, $\mathrm{div}\, B = 0$; see [14, 13].

The form ω_F is called the *magnetic symplectic form.* We show how this form generates the Weyl–type associative product $\underset{F}{\star}$ of functions over the phase space. The q^j, p_k coordinates on this space correspond to the position of the charged particle and its gauge invariant kinetic momentum. The commutation relations between the corresponding quantum operators $\hat{q}^j, \hat{p}_k$ are the following

$$[\hat{q}^j, \hat{q}^k] = 0, \qquad [\hat{q}^j, \hat{p}_k] = i\hbar\delta^j_k, \qquad [\hat{p}_j, \hat{p}_k] = i\hbar F_{kj}(\hat{q}). \tag{1.3}$$

The usual realization of these operators in the Hilbert space $L^2(\mathbb{R}^n_q)$ is $\hat{q} = q$, $\hat{p} = -i\hbar\partial/\partial q - (e/c)\Phi(q)$, where $(e/c)d(\Phi dq) = F$, and Φ is the gauge potential. Specifically, for $n = 3$:

$$\Phi = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3), \qquad \mathrm{curl}\mathcal{A} = B, \qquad \hat{p}_j = -i\hbar\frac{\partial}{\partial q_j} - \frac{e}{c}\mathcal{A}_j(q) \quad (j = 1, 2, 3); \tag{1.4a}$$

for $n = 4$:

$$\Phi = (-a, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3), \qquad \mathrm{curl}\mathcal{A} = B, \qquad -\Big(\frac{1}{c}\frac{\partial\mathcal{A}}{\partial t} + \frac{\partial a}{\partial q}\Big) = E,$$

$$\hat{p}_0 = -\frac{i\hbar}{c}\frac{\partial}{\partial t} + \frac{e}{c}a(t, q), \quad \hat{p}_j = -i\hbar\frac{\partial}{\partial q_j} - \frac{e}{c}\mathcal{A}_j(t, q) \quad (j = 1, 2, 3). \tag{1.4b}$$

All the formulae obtained in the paper depend on the symplectic form (1.1) only, but not on the choice of potentials, and so, all results are gauge independent.

The noncommutative product $\underset{F}{\star}$ which we construct necessarily reproduces the commutation relations (1.3) on coordinate functions:

$$q^j \underset{F}{\star} q^k - q^k \underset{F}{\star} q^j = 0, \qquad q^j \underset{F}{\star} p_k - q^j \underset{F}{\star} p_k = i\hbar\delta^j_k, \qquad p_j \underset{F}{\star} p_k - p_k \underset{F}{\star} p_j = i\hbar F_{kj}(q).$$

The operators of left multiplication $q\underset{F}{\star}$ and $p\underset{F}{\star}$ (as well as right multiplication) are derived without difficulty from these relations, using the noncommutative calculus [15, 16]; the result incorporates Valatin's [17] primitive of the closed 2-form F (see Section 2). We note that the problem of a finding a gauge invariant symbol product in a convenient closed form was first addressed by Stratonovich [18].

In Section 3 we derive general representations for $\underset{F}{\star}$ in two equivalent forms, both valid when $F \neq 0$. The first is analogous to the exponential Janus derivative representation [19, 20, 21] due to Groenewold. The second is a modification of the ($F = 0$) Berezin's integral for the Weyl product. In both versions there appears an additional electromagnetic action (flux) over triangles in phase space. Moreover, the Groenewold–like product formula admits generalization for N multipliers via the magnetic area of polygon membranes.

Formulas for the $\underset{F}{\star}$ product can be easily expanded to obtain formal $\hbar \to 0$ power series. Higher order terms beyond the Poisson bracket contribution are functions of derivatives of F_{jk}. This series coincides structurally with that recently obtained [22] by Müller.

The integral formula for $\underset{F}{\star}$ is also related to constructions [23, 24, 25] of the Wigner function in the presence of electromagnetic fields. In this way in Section 4 we show that the $\underset{F}{\star}$ product can be produced by a convolution over $T\mathbb{R}^n$. This convolution is generated by a version of the Connes' tangential groupoid [26, 27, 28] but with an additional rapidly oscillating factor represented by the electromagnetic flux. Because of the rapid oscillations, this type of star-product is outside the framework of the formal deformation quantization method.

In Section 5 we analyze, following the general approach of [29], the structure of the symplectic groupoid corresponding to $\underset{F}{\star}$. This structure is given on the secondary cotangent bundle $T^*(T^*\mathbb{R}^n) = T^*\mathbb{R}^n \oplus \mathbb{R}^{2n}$. The first cotangent bundle, $T^*\mathbb{R}^n = \mathbb{R}^n_q \otimes \mathbb{R}^n_p$, is the primary phase space, over which we construct the product $\underset{F}{\star}$. We show how the symplectic groupoid structure on $T^*(T^*\mathbb{R}^n)$ senses the magnetic correction F in the symplectic form, and how the space $\mathbb{R}^{2n}$, dual to $T^*\mathbb{R}^n$, is equipped with a pseudogroup structure controlled by the Lorentz momentum of membranes.

From this point of view, we claim that in the formula for $\underset{F}{\star}$ it would be more natural to consider not the usual geodesic triangle, but the triangle with three additional "wings" directed vertically (i.e., parallel to the p-direction) in the phase space. The shape of wings is determined by the symplectic groupoid structure.

Then in Section 6 we investigate what happens if the Weyl ordering of noncommuting coordinates is changed to some other ordering. The wide (matrix) family of orderings introduced in [30] we relate to phases in the exponential representation of the $*$-product. These phases can be again presented as symplectic areas of membranes. The membranes are combinations of the basic triangle with additional wings that are now not necessarily vertical. The shape and direction of the wings exactly control the choice of the ordering in quantization, and again it is related to the symplectic groupoid structure over the secondary phase space $T^*(T^*\mathbb{R}^n)$.

The symplectic area of the wings give three additional contributions to the phase. The transform from the original Berezin phase (the Weyl case) to the new one, generated by the wings, can be considered as a type of gauge transformation of the "symplectic potential". On the level of $*$-products this is the transformation from the distinguished Weyl choice to other ordering choices. In a sense this is the "gauge" of quantization.

The special features of the Weyl quantization which have made it the preferred choice [2, 32, 20, 33, 21] for physical applications are: 1) it treats $\hat{q}$ and $\hat{p}$ symmetrically; 2) self-adjoint operators have real symbols; and, 3) the Groenewold–Moyal bracket is an even function of $\hbar$, in particular its leading semiclassical correction is $O(\hbar^2)$, not $O(\hbar)$. From the symplectic point of view, the Weyl ordering seems distinguished since the corresponding membranes are of the simplest shape (no wings).

The Wick normal and anti-normal orderings $\overset{2}{\hat{z}^*}, \overset{1}{\hat{z}}$ and $\overset{1}{\hat{z}^*}, \overset{2}{\hat{z}}$ (where $z = q + ip$) correspond to pure imaginary wings of membranes in the product formulas.

Other convenient orderings — the standard $\overset{2}{\hat{q}}, \overset{1}{\hat{p}}$ and anti-standard $\overset{1}{\hat{q}}, \overset{2}{\hat{p}}$ — correspond to the case when the wings are parallel to the basic triangle and the total membrane becomes a plane rectangle. These standard and anti-standard cases correspond to the push- and pull-groupoid structure on $T\mathbb{R}^n$ (see Section 4). From the symplectic point of view these cases are singular because of the totally caustic character of the graph of symplectic groupoid multiplication corresponding to these cases.

In the last sections 7 and 8 we apply these ideas to the quantum dynamical problem: a charged particle in an electromagnetic field. The basic results for this system were found by Dirac, Fock, Peierls, Schwinger. Gauge invariant versions of the WKB approximation were developed [31, 34, 35, 36, 37, 38] mostly in terms of integral kernels (Green functions). In the context of the present paper we can use the $\underset{F}{\star}$ symbol calculus to obtain a phase space gauge invariant treatment of this problem.

We first study, in Section 7, the pure magnetic situation without electric field. We represent the gauge invariant version of the quantum evolution equation over phase space and show how the Marinov phase and summation rule are generalized in the presence of the magnetic field. Also we investigate how the formulae for semiclassical solutions sense the generic wings of the membranes (i.e., arbitrary, not Weyl ordering).

At the end of Section 7, using the membrane generalization of the Groenewold formula, we represent the symbol of the evolution operator exactly in a continual form. This continual membrane formula is dual to the Feynman path integral representation.

Then, in Section 8, we consider a time-dependent electromagnetic field and represent the gauge invariant quantum equations over phase space in both the nonrelativistic and relativistic cases. Here we use dynamical quantum products which are time-dependent. The evolution of commutation relations in time is controlled by the electric field.

We describe the semiclassical solution of the Cauchy problem using membranes in 7-dimensional contact space $\mathbb{R}_t \times \mathbb{R}^3_q \times \mathbb{R}^3_p$. The boundary of these dynamic membranes are given by the solution of two classical systems: one for the given particle and an additional one for a "virtual" particle of infinite mass.

2 Magnetic product for Weyl ordering

We begin with the definition of magnetic product on a function space over $\mathbb{R}^{2n}$. The logic is the following: we transform relations (1.3) to the standard Heisenberg relations, apply the standard Weyl operators of regular representation, and then transform back to the magnetic variables. As the result, we obtain a formula for magnetic product in terms of left and right regular representation of the algebra (1.3).

First we recall the properties of closed forms on $\mathbb{R}^n$.

Lemma 1 *Let $F = \frac{1}{2}F_{jk}(q)dq^k \wedge dq^j$ be a closed 2-form on $\mathbb{R}^n$. Consider the vector-valued 1-form Fdq with components $(F(q)\,dq)_j = F_{jk}(q)\,dq^k$ and define the two point vector potential*

$$A(q,q') = \frac{1}{|q-q'|}\int_{q'}^{q} |\tilde{q}-q'|F(\tilde{q})d\tilde{q}, \tag{2.1}$$

where the integral is taken along the straight line path from q' to q, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Then for an arbitrary fixed $q' \in \mathbb{R}^n$ the 1-form $A(q,q')dq$ is a primitive of F:

$$d(A\,dq) = F.$$

The choice of a primitive (2.1) (*gauge choice*) *is uniquely characterized by the orthogonality condition*

$$A(q, q') \cdot (q - q') = 0. \tag{2.2}$$

Note that construction (2.1) is different from that usually used in proofs of the known Poincaré lemma in the theory of differential forms. On the other hand, (2.1) is just a simple particular case of the solution of Lie system related to a general Poisson bracket; (see [16] page 81 and references therein). Formula (2.1) was obtained by Valatin [17] for electromagnetic tensors. The characteristic condition (2.2) was stressed by Dirac (see [17] p. 101).

We need several other properties of Valatin's primitive.

Lemma 2 *The following formulas hold*:

$$d_{q'}(A(q, q')\, dq) = d_q(A(q', q)\, dq'),$$

$$A(q, q') - A(q', q) = \int_{q'}^{q} F(\tilde{q})\, d\tilde{q},$$

$$A(q, q') + A(q', q) = \frac{1}{|q - q'|} \int_{q_m}^{q} |\tilde{q} - \tilde{q}^*| \big(F(\tilde{q}) - F(\tilde{q}^*)\big)\, d\tilde{q},$$

where the integrals are taken along the straight line paths, and $\tilde{q}^* = 2q_m - \tilde{q}$ *is the point symmetric to* $\tilde{q}$ *with respect to the middle point* $q_m = \frac{1}{2}(q + q')$.

Let $\triangle$ be the triangle in $\mathbb{R}^n$ with vertices q, q', q''. Consider the integral (flux) of the form F:

$$\mathrm{Flux}_{\,q''}(q, q') \equiv \int_{\triangle} F. \tag{2.3}$$

Note that here and everywhere in the sequel the orientation of a membrane corresponds to the sequence of its vertices (or sides) read from right to left; so the orientation of $\triangle$ corresponds to the sequence $q \leftarrow q' \leftarrow q''$.

Lemma 3 *The following formulas hold*:

$$\mathrm{Flux}_{\,q''}(q, q') = \int_{q'}^{q} A(\tilde{q}, q'') d\tilde{q}, \tag{2.4}$$

$$\frac{\partial}{\partial q} \mathrm{Flux}_{\,q''}(q, q') = A(q, q'') - A(q, q'),$$

where the integral in (2.4) *is taken along the straight line path.*

Now let us fix the second argument of $A(q, q')$ at some point, say, $q' = 0$ and introduce the operators

$$\hat{p}' = \hat{p} + A(\hat{q}, 0).$$

Since $\hat{q}$, $\hat{p}$ satisfy relations (1.3), the new set of operators $\hat{q}$, $\hat{p}'$ satisfy the standard Heisenberg commutation relations

$$[\hat{q}^j, \hat{p}'_k] = i\hbar\delta^j_k, \quad [\hat{q}^j, \hat{q}^k] = [\hat{p}'_j, \hat{p}'_k] = 0. \tag{2.5}$$

Any Weyl-symmetrized function of operators $\hat{q}, \hat{p}$ can be transformed to a function of operators $\hat{q}, \hat{p}'$ by the formula

$$f(\hat{q}, \hat{p}) = f\left(\frac{\overset{3}{\hat{q}} + \overset{1}{\hat{q}}}{2}, \overset{2}{\hat{p}'} - \tilde{A}(\overset{3}{\hat{q}}, \overset{1}{\hat{q}})\right), \tag{2.6}$$

where

$$\tilde{A}(q, q') \equiv \int_0^1 A(q\mu + q'(1-\mu), 0)d\mu.$$

We have used here formulas of noncommutative analysis [15] and [16] pp. 277–295; the superscripts on top of operators denote the order of application.

Throughout the paper we will not give a characterization of the spaces the symbols must belong to in order that (2.6) and the subsequent product formulas are well defined. This is a separate technical (and often not simple) question which has been extensively investigated in the pseudodifferential operator literature [39, 40, 30, 16]. The reader can consider all formulas as formally algebraic or, depending on the formula, assume an appropriate simple symbol class such as polynomials, smooth rapidly decreasing functions, etc.

For the Heisenberg algebra (2.5) the operators of left and right regular representation are well known. Namely, consider arbitrary Weyl-symmetrized function g' with operator arguments $\hat{q}, \hat{p}'$:

$$\hat{g}' \equiv g'(\hat{q}, \hat{p}') = g'\left(\frac{\overset{3}{\hat{q}} + \overset{1}{\hat{q}}}{2}, \overset{2}{\hat{p}'}\right). \tag{2.7}$$

Then the following left and right multiplication formulas hold [39, 16, 30, 41]:

$$\begin{aligned}
\hat{q}\hat{g}' &= \widehat{L'_q g'}, & L'_q &= q + \tfrac{1}{2}i\hbar\partial_{p'}; \\
\hat{p}'\hat{g}' &= \widehat{L'_p g'}, & L'_p &= p' - \tfrac{1}{2}i\hbar\partial_q; \\
\hat{g}'\hat{q} &= \widehat{R'_q g'}, & R'_q &= q - \tfrac{1}{2}i\hbar\partial_{p'}; \\
\hat{g}'\hat{p}' &= \widehat{R'_p g'}, & R'_p &= p' + \tfrac{1}{2}i\hbar\partial_q.
\end{aligned} \tag{2.8}$$

Here we denote $\partial_q \equiv \partial/\partial q$ and $\partial_{p'} = \partial/\partial p'$. Operators L', R' satisfying multiplication formulas (2.8) are called the left and right regular representation of the given algebra, in our case, the algebra (2.5).

On the right-hand side of the formula (2.6) operators $\overset{1}{\hat{q}}$ and $\overset{3}{\hat{q}}$ in arguments of $\tilde{A}$ can be considered as multiplication by $\hat{q}$ from the left and from the right, and so they can be replaced by L'_q and R'_q acting on arguments of f. Thus we obtain from (2.6)

$$f(\hat{q},\hat{p}) = f'\left(\frac{\overset{3}{\hat{q}} + \overset{1}{\hat{q}}}{2}, \overset{2}{\hat{p}'}\right) = f'(\hat{q},\hat{p}'),$$

where

$$f'(q,p') = f(q, p' - \tilde{A}(\overleftarrow{L'_q}, \overleftarrow{R'_q})) = \exp\{-\tilde{A}(L'_q, R'_q)\partial_{p'}\} f(q,p'),$$

and the left arrows mean that operators act on arguments standing to their left. After substitution of explicit formulas for L'_q, R'_q from (2.8) we conclude

$$f'(q,p') = \exp\{-\tilde{A}(q + \tfrac{1}{2}i\hbar\partial_{p'}, q - \tfrac{1}{2}i\hbar\partial_{p'})\partial_{p'}\} f(q,p').$$

Note that by Lemma 2

$$\tilde{A}(q + u/2, q - u/2)u = \int_{q-u/2}^{q+u/2} A(\tilde{q}, 0)d\tilde{q} = \mathrm{Flux}_0(q + u/2, q - u/2).$$

So we obtain the transformation formula

Proposition 1 *Any Weyl-symmetrized function f in operators $\hat{q},\hat{p}$, satisfying commutation relations* (1.3) *, can be transformed to the Weyl-symmetrized function f' in operators $\hat{q},\hat{p}'$ satisfying Heisenberg relations* (2.5)*. This transform is given by formula*

$$f' = U_F f, \quad U_F = \exp\left\{\frac{i}{\hbar}\,\mathrm{Flux}_0(q + \tfrac{1}{2}i\hbar\partial_p, q - \tfrac{1}{2}i\hbar\partial_p)\right\}, \tag{2.9}$$

where the Flux_0 *was defined in* (2.3)*.*

Using this transform one easily obtains all objects which are needed for the algebra (1.3). For instance, the operators of left and right regular representation for (1.3) are the following

$$\begin{aligned} L_q &= U_F^{-1} \cdot L'_q \cdot U_F, & R_q &= U_F^{-1} \cdot R'_q \cdot U_F, \\ L_p &= U_F^{-1} \cdot L'_p \cdot U_F - A(L_q, 0), & R_p &= U_F^{-1} \cdot R'_p \cdot U_F - A(R_q, 0). \end{aligned}$$

Applying explicit formulas (2.8) for L', R', (2.9) for U_F and using Lemma 3 we get

$$\begin{aligned} L_q &= q + \tfrac{1}{2}i\hbar\partial_p, & R_q &= q - \tfrac{1}{2}i\hbar\partial_p, \\ L_p &= p - \tfrac{1}{2}i\hbar\partial_q - A(L_q, R_q), & R_p &= p + \tfrac{1}{2}i\hbar\partial_q - A(R_q, L_q). \end{aligned} \tag{2.10}$$

Lemma 4 *Operators L_q, L_p satisfy commutation relations* (1.3)*, and operators R_q, R_p satisfy the conjugate relations* (*with opposite signs*)*. Operators L commute with R.*

If we know the regular representation of commutation relations, we know the product of symbols (see [16] Appendix 2).

Theorem 1 *The product of Weyl-symmetrized functions in operators satisfying relations* (1.3) *is given by*

$$f(\hat{q},\hat{p})\cdot g(\hat{q},\hat{p}) = (f \underset{F}{\star} g)(\hat{q},\hat{p}), \tag{2.11}$$

where $\underset{F}{\star}$ *is the associative product of functions over* $\mathbb{R}^{2n}$ *defined by*

$$f \underset{F}{\star} g = f(L_q, L_p)g = g(R_q, R_p)f.$$

We call $\underset{F}{\star}$ the *magnetic product* of Weyl type. In particular, we have $L_q = q\underset{F}{\star}$, $L_p = p\underset{F}{\star}$, and $R_q = \underset{F}{\star}q$, $R_p = \underset{F}{\star}p$. These multiplication operators satisfy relations (1.3) and their conjugate companions; left and right multiplications commute with each other.

3 Exponential formula for magnetic product

The next useful stage in evaluation of the $\underset{F}{\star}$-product is to bring it into an exponential form. First, we apply Proposition 1 and the usual Groenewold formula [19, 39] known for Weyl product of symbols over the Heisenberg algebra, namely,

$$f' \star g' = f' \exp\Big\{ -\frac{i\hbar}{2}\overleftarrow{\partial}\, J^{-1}\overrightarrow{\partial}\Big\} g' = f'(q+\overrightarrow{u}/2, p)\; g'(q-\overleftarrow{u}/2, p), \tag{3.1}$$

where

$$\overleftarrow{u} = i\hbar\overleftarrow{\partial_p}, \quad \overrightarrow{u} = i\hbar\overrightarrow{\partial_p}, \quad \partial = (\partial_q, \partial_p).$$

From (2.9) we calculate

$$\begin{aligned} U_F f \star U_F g \ &= f \exp\Big\{ \frac{i}{\hbar}\,\mathrm{Flux}_0(q + (\overleftarrow{u} + \overrightarrow{u})/2, q + (\overrightarrow{u} - \overleftarrow{u})/2) \\ &\quad + \frac{i}{\hbar}\,\mathrm{Flux}_0(q + (\overrightarrow{u} - \overleftarrow{u})/2, q - (\overleftarrow{u} + \overrightarrow{u})/2) - \frac{i\hbar}{2}\overleftarrow{\partial}\, J^{-1}\overrightarrow{\partial}\Big\}\, g. \end{aligned}$$

Applying the inverse transformation U_F^{-1} we obtain

$$\begin{aligned} U_F^{-1}(U_F f \star U_F g) = \ & f \exp\Big\{ -\frac{i}{\hbar}\,\mathrm{Flux}_0(q + (\overleftarrow{u} + \overrightarrow{u})/2, q - (\overleftarrow{u} + \overrightarrow{u})/2) \\ & + \frac{i}{\hbar}\,\mathrm{Flux}_0(q + (\overleftarrow{u} + \overrightarrow{u})/2, q + (\overrightarrow{u} - \overleftarrow{u})/2) \\ & + \frac{i}{\hbar}\,\mathrm{Flux}_0(q + (\overrightarrow{u} - \overleftarrow{u})/2, q - (\overleftarrow{u} + \overrightarrow{u})/2) - \frac{i\hbar}{2}\overleftarrow{\partial}\, J^{-1}\overrightarrow{\partial}\Big\}\, g. \end{aligned}$$

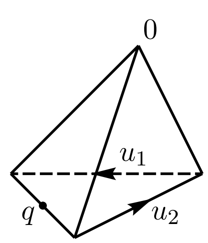

Figure 1.

The sum of three fluxes which appeared at last exponent can be simplified if we look at the geometrical picture. Indeed, the three fluxes represent integrals of the form F over three triangles with common vertex 0. They are sides of the tetrahedron. By the Stokes theorem these three fluxes together are equal to the flux over the bottom triangle; see Fig. 1, in which the vectors u_1 and u_2 represent the operators $\overrightarrow{u}$ and $\overleftarrow{u}$, respectively.

The relation

$$f \underset{F}{\star} g = U_F^{-1}(U_F f \star U_F g)$$

and the representation of its right hand side which we have derived above generate the following statement.

Proposition 2 *The magnetic product* $\underset{F}{\star}$, *corresponding to commutation relations* (1.3), *can be calculated by the formula*

$$(f \underset{F}{\star} g)(q,p) = f(q,p) \exp\left\{ \frac{i}{\hbar}\phi(q, i\hbar\overleftarrow{\partial_p}, i\hbar\overrightarrow{\partial_p}) + \frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\,\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\,\overrightarrow{\partial_q}\right)\right\} g(q,p). \tag{3.2}$$

Here

$$\phi(q, u_2, u_1) = \int_{\triangle_q(u_2,u_1)} F, \tag{3.3}$$

and $\triangle_q(u_2, u_1)$ *is a membrane in* $\mathbb{R}^n$ *whose boundary is the triangle constructed by the middle point* q *of one side and by the two other sides* u_2, u_1. *The magnetic flux* (3.3) *can also be represented in the form*

$$\phi = \int_0^1 d\mu \int_0^\mu d\nu \; u_2 F\big(q + (\mu - \tfrac{1}{2})u_1 + (\nu - \tfrac{1}{2})u_2\big)u_1.$$

Note the differential operators in the exponent (3.2) act only on the arguments of target functions f and g, but not on the argument q in the flux ϕ.

Also note that operators $i\hbar\partial_p$, which we substitute into the flux ϕ in the exponent (3.2), are of order $O(\hbar)$ over the space of non-oscillating functions as $\hbar \to 0$. So the flux ϕ actually is of order $O(\hbar^2)$, and the right hand side of (3.2) can be easily expanded as a power series in $\hbar$. Thus for non-oscillating (as $\hbar \to 0$) functions f, g

$$f \underset{F}{\star} g \simeq \sum_{|\gamma|,|\varepsilon|=0}^{\infty} \frac{(-1)^{|\varepsilon|}}{\gamma!\varepsilon!}\Big(-\frac{i\hbar}{2}\Big)^{|\gamma|+|\varepsilon|}$$

$$\times\, \partial_q^\varepsilon \partial_p^\gamma f \exp\left\{ -\frac{i\hbar}{2} \sum_{|\alpha|,|\beta|=0}^{\infty} \frac{c_{|\alpha|,|\beta|}}{\alpha!\beta!}\Big(-\frac{i\hbar}{2}\Big)^{|\alpha|+|\beta|} (\overleftarrow{\partial_p})^\alpha \langle \overleftarrow{\partial_p}, \partial^{\alpha+\beta}F(q)\overrightarrow{\partial_p}\rangle (\overrightarrow{\partial_p})^\beta \right\} \partial_q^\gamma \partial_p^\varepsilon g,$$

where

$$c_{s,m} = \frac{e[m+1]}{(s+1)(m+1)} - \frac{e[s+m]}{(s+1)(s+m+2)},$$

and $e[l] = 0$ if l is even, and $e[l] = 1$ if l is odd. Here $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^n$ and the greek letters are multi-indices.

Of course, the first two terms of this expansion are

$$f \underset{F}{\star} g = fg - \frac{i\hbar}{2}\{f,g\}_F + O(\hbar^2),$$

where $\{,\}_F$ is the Poisson bracket on $\mathbb{R}^{2n}$ corresponding to the symplectic form ω_F, i.e.,

$$\{f,g\}_F = \partial_p f \partial_q g - \partial_q f \partial_p g + \langle \partial_p f, F(q)\partial_p g\rangle. \tag{3.4}$$

In the case where F is restricted to be the pure magnetic form (1.2a), then the $\hbar$ series above is equivalent to the one derived in [22].

The formula (3.2) is still not presented in a completely symplectic manner. We have there the flux ϕ, which is the integral of the form F over the triangle in q-space. The other part of the exponent in (3.2) can also be related to the area of a triangle, but in the (q,p)-space; see Fig. 2

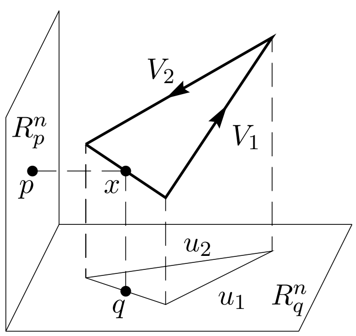

Figure 2.

This triangle $\Sigma_x(V_2,V_1)$ is constructed by the middle point x of one side, and by the two opposite sides V_2, V_1. Its projection onto $\mathbb{R}^n_q$ coincides with the triangle $\triangle_q(u_2,u_1)$.

Theorem 2 *The magnetic product can be represented in the form*

$$(f \underset{F}{\star} g)(x) = \exp\left\{\frac{i}{\hbar}\int_{\Sigma_x(\widehat{V}_2,\widehat{V}_1)} \omega_F\right\} f(x_2)g(x_1)\Big|_{x_1=x_2=x}, \tag{3.5}$$

where ω_F is the magnetic symplectic form (1.1), $\widehat{V} = (i\hbar\partial_p, i\hbar\partial_q)$, *and* $x=(q,p)\in\mathbb{R}^{2n}$.

In (3.5) one has to evaluate first the symplectic ω_F-area of the membrane $\Sigma_x(V_2,V_1)$, and then substitute the operators $\widehat{V}_1,\widehat{V}_2$ for the vectors V_1,V_2; the operator $\widehat{V}_1$ is applied to the argument x_1 and $\widehat{V}_2$ to the argument x_2.

Actually, formula (3.5) may be immediately generalized to the case of several multipliers. Consider a membrane in $\mathbb{R}^{2n}$ whose boundary is formed by N vectors $V_1,V_2,\dots,V_N$ as sides of a polygon, and by additional point x as a middle point of the $(N+1)$th side. Denote this membrane by $\Sigma_x(V_N,\dots,V_1)$; see Fig. 3.

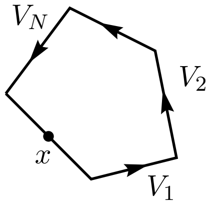

Figure 3.

Corollary 1 *The following formula holds*:

$$(f_N \underset{F}{\star} \dots \underset{F}{\star} f_1)(x) = \exp\left\{\frac{i}{\hbar}\int_{\Sigma_x(\widehat{V}_N,\dots,\widehat{V}_1)} \omega_F\right\} f_N(x_N)\cdots f_1(x_1)\Big|_{x_1=\dots=x_N=x}. \tag{3.6}$$

An immediate consequence of (3.5) is an integral formula for $f \underset{F}{\star} g$. Indeed, in (3.5) we have a pseudodifferential operator acting on f and g. By the usual formulas via Fourier and inverse Fourier transform we easily calculate this action and obtain an integral representation for $f \underset{F}{\star} g$. The triangle $\Sigma_x(V_2, V_1)$ in this new formula will be described not by the sides V_2, V_1, but by the middle points x_2, x_1 of those sides. Such a triangle in $\mathbb{R}^{2n}$, constructed by three middle points x, x_2, x_1, we denote by $\Sigma(x, x_2, x_1)$.

Proposition 3 *The magnetic product of the Weyl type is given by the integral formula*

$$(f \underset{F}{\star} g)(x) = \frac{1}{(\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \exp\left\{\frac{i}{\hbar}\int_{\Sigma(x,x_2,x_1)} \omega_F\right\} f(x_2) g(x_1)\, dx_2\, dx_1\,. \tag{3.7}$$

Actually, the magnetic part F of the form ω_F is integrated in (3.5) and (3.7) over the projection of $\Sigma(x, x_2, x_1)$ onto q-space.

In the case $F = 0$, (3.7) becomes the Berezin formula for the Weyl product, and (3.6) becomes equivalent to the N-factor Weyl products obtained in [20, 21].

4 Tangential groupoid and magnetic cocycle

Formula (3.7) for magnetic noncommutative product over $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ can also be derived from the Connes type tangential groupoid structure on $T\mathbb{R}^n$ [26, 27, 28], equipped with an additional magnetic phase factor (cocycle).

The space $T\mathbb{R}^n$ consists of pairs (q, u), where $q \in \mathbb{R}^n$ and $u \in T_q\mathbb{R}^n$. Vector u is interpreted as a "replacement" at point q. The first and simplest groupoid structure on $T\mathbb{R}^n$ is given by formula $(q_2, u_2) \circ (q_1, u_1) = (q_1, u_2 + u_1)$ iff $q_1 = q_2$. This groupoid is commutative; it is called the *Galileo groupoid*. This structure is shown in Fig. 4. The set of units of this groupoid consists of all pairs $(q, 0)$; so, the set of units is the configuration space $\mathbb{R}^n_q$ considered as the zero-section in $T\mathbb{R}^n$.

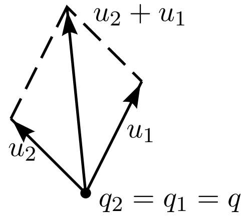

Figure 4.

Other geometric combinations of the vectors u_1, u_2 and the points q_1, q_2 may be assembled to give noncommutative groupoids. Three simple possibilities are shown in Fig. 5.

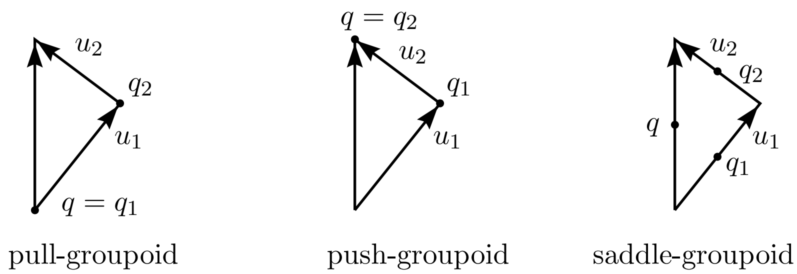

Figure 5.

The multiplication rule, for instance, in the *pull-groupoid* is $(q_2, u_2) \circ (q_1, u_1) = (q_1, u_2 + u_1)$ iff $q_1 + u_1 = q_2$. For the *saddle-groupoid* one has

$$(q_2, u_2) \circ (q_1, u_1) = (q, u_2 + u_1) \quad \text{iff} \quad q_1 + u_1/2 = q_2 - u_2/2.$$

In this case, q is the middle point of the third side of the triangle: $q = q_1 + u_2/2 = q_2 - u_1/2$.

All these groupoids are specific cases of a general τ-*groupoid*, where $0 \le \tau \le 1$. Pull-, push-, and saddle-cases correspond to $\tau = 1$, $\tau = 0$, and $\tau = \frac{1}{2}$, respectively. In Fig. 6 one can see the generic case corresponding to some τ $(\frac{1}{2} < \tau < 1)$; in this case $q_2 = q_1 + \tau u_1 + (1 - \tau) u_2$ and $q = q_1 + (1 - \tau) u_2$. The set of units for all of these groupoids is $\mathbb{R}^n_q \subset T\mathbb{R}^n$.

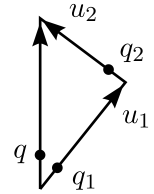

Figure 6.

On any (measurable) groupoid there is a convolution of distributions [26, 42]

$$(\phi_2 \odot \phi_1)(a) = \int_{a = b \circ c} \phi_2(b) \phi_1(c) \, d\mu_b(c), \tag{4.1}$$

where $d\mu_b$ is the Haar measure on fibres of the left groupoid mapping $c \to c \circ c^{-1}$. For example, in the case of saddle-groupoid we have

$$(\phi_2 \odot \phi_1)(q,u) = \int_{\mathbb{R}^n} \phi_2(q + u_1/2, u - u_1)\phi_1(q - (u - u_1)/2, u_1) du_1. \tag{4.2}$$

Each convolution over $T\mathbb{R}^n$ generates a noncommutative product over $T^*\mathbb{R}^n$ just by Fourier transform between u and p coordinates:

$$f^{\sim}(q,u) \equiv \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} e^{iup/\hbar} f(q,p)\, dp = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iuk} f(q,\hbar k)\, dk. \tag{4.3}$$

For example, if one takes the saddle-groupoid convolution $\odot$ (4.2) then the corresponding product over $T^*\mathbb{R}^n$ is the usual Groenewold-Moyal product $\star$ (3.1), i.e.,

$$(f \star g)^{\sim} = f^{\sim} \odot g^{\sim}. \tag{4.4}$$

Remark 1 In this approach the quantum convolution $\odot$ is not an $\hbar$-deformation of a commutative one. The deformation parameter $\hbar$ appears only in the Fourier transform (4.3); this is what makes it possible to include into the quantization scheme functions over $T^*\mathbb{R}^n$ that oscillate rapidly as $\hbar \to 0$. For non-oscillating functions $f(q,p)$ the transform (4.3) admits the classical limit as $\hbar \to 0$; here one obtains the correspondence $f(q,p) \to f(q,0)\delta(u)$, $\hbar \to 0$ between the classical commutative algebra of functions over the phase space $T^*\mathbb{R}^n$ and the commutative subalgebra of distributions over $T\mathbb{R}^n$ of the type $\varphi(q)\delta(u)$ concentrated at the zero "replacement" $u = 0$. Any replacement by a vector u_0, represented over $T\mathbb{R}^n$ as the distribution $\delta(u - u_0)$, after the transform inverse to (4.3) generates the phase space observable $f(q,p) = \exp\{-iu_0 p/\hbar\}$. The corresponding operator $\hat{f}$ (if the magnetic tensor $F = 0$) is just the Heisenberg translation operator $\hat{f}\psi(q) = \psi(q - u_0)$. This is the simplest example of rapidly oscillating functions f which one has to include into algebra of observables over $T^*\mathbb{R}^n$, but which lie outside the formal deformation quantization approach; see also [27, 43].

Now let us return to the relation (4.4) between star-product and convolution. The question arises: how to get the magnetic product $\underset{F}{\star}$ (3.7) via the groupoid structure on $T\mathbb{R}^n$? Let us slightly generalize (4.1) by introducing an exponential phase factor:

$$(\phi_2 \odot \phi_1)(a) = \int_{a = b \circ c} e^{i\Phi(b,c)} \phi_2(b)\phi_1(c)\, d\mu_b(c). \tag{4.5}$$

Here Φ is a groupoid cocycle, i.e.,

$$\Phi(d, b \circ c) - \Phi(d \circ b, c) + \Phi(b,c) - \Phi(d,b) = 0, \quad \Phi(b,c) = -\Phi(c^{-1}, b^{-1})$$

for any $b, c, d \in T\mathbb{R}^n$. The cocycle Φ is called the coboundary iff

$$\Phi(b,c) = \psi(b) + \psi(c) - \psi(b \circ c), \quad \psi(b^{-1}) = -\psi(b).$$

If Φ in (4.5) is a coboundary, then convolutions (4.5) and (4.1) are actually equivalent.

In the case of saddle-groupoid we can take the following "magnetic" cocycle (coboundary):

$$\Phi = \frac{1}{\hbar} \int_{\triangle(q,q_2,q_1)} F, \tag{4.6}$$

where F is the Faraday 2-form over $\mathbb{R}^n$, and $\triangle(q, q_2, q_1)$ is the triangle with middle points q, q_2, q_1. In this case the coboundary function ψ at the point $b = (q, u)$ is given by the integral $\psi(b) = \frac{1}{\hbar} \int \widetilde{A}\, d\tilde{q}$ along the chord $[q - \frac{1}{2}u, q + \frac{1}{2}u]$. Note that ψ controls the additional phase factor in the gauge invariant version of the Wigner function [24, 25].

Proposition 4 *Let* $\underset{F}{\odot}$ *be the saddle-groupoid convolution, equipped with the magnetic cocycle* (4.6)*. Then the Fourier transform* (4.3) *relates this convolution to the magnetic product* $\underset{F}{\star}$ *via*

$$(f \underset{F}{\star} g)^{\sim} = f^{\sim} \underset{F}{\odot} g^{\sim}.$$

Note that other types of groupoids (like pull-, push-) will also generate some magnetic products over $\mathbb{R}^{2n} = T^*\mathbb{R}^n$. In Section 6 we will consider a variety of such products. On the other hand, we will show that not all of products over $T^*\mathbb{R}^n$ are generated from the convolution over $T\mathbb{R}^n$ in this fashion.

In Section 5 we analyze the magnetic product (3.7) from the view point of symplectic groupoid structure of the secondary phase space (on the secondary cotangent) $T^*(T^*\mathbb{R}^n)$. Regarding this we mention the following.

Proposition 5 *Each groupoid structure on* $T\mathbb{R}^n$ *whose set of units is* $\mathbb{R}^n_q$*, uniquely determines a symplectic groupoid structure on* $T^*(T^*\mathbb{R}^n)$*. Here* $T^*\mathbb{R}^n$ *is equipped with the symplectic form* $\omega_0 = dp \wedge dq$*. If, in addition, on* $T\mathbb{R}^n$ *a cocycle of type* (4.6) *is given, then on* $T^*(T^*\mathbb{R}^n)$ *we have a symplectic groupoid structure corresponding to the magnetic form* ω_F*.*

5 Symplectic groupoid and membranes with wings

Formula (3.7) represents the exact magnetic product. On the other hand, it is known [16], in a very general context, how to construct quantum products of functions over an arbitrary Poisson manifold $\mathcal{N}$ in the semiclassical approximation, to all orders in $\hbar \to 0$. This approximate product takes the form

$$(f * g)(x) \simeq \int\int K_\hbar(x, x_2, x_1) f(x_2) g(x_1)\, dx_1\, dx_2. \tag{5.1}$$

Here $x, x_2, x_1 \in \mathcal{N}$, and $K_\hbar$ is a "wave function" corresponding to some Lagrangian submanifold Λ_* in the "phase space" $\mathcal{E} \times \mathcal{E} \times \mathcal{E}$, where $\mathcal{E}$ is the symplectic groupoid over $\mathcal{N}$. In our case $\mathcal{N} = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ with Poisson bracket (3.4). The Lagrangian submanifold Λ_* is the graph of groupoid multiplication in $\mathcal{E}$. If this graph is one-to-one projected onto the "configuration" space $\mathcal{N}\times\mathcal{N}\times\mathcal{N}$ along the polarization then the function $K_\hbar$ is just a WKB function

$$K_\hbar = \exp\{iS/\hbar\}\varphi + O(\hbar) \tag{5.2}$$

whose phase S is the Poincaré-Cartan action on Λ_* and φ is the solution of the corresponding transport equation.

In specific cases, for instance, in our case $\mathcal{N}=\mathbb{R}^{2n}$ with bracket (3.4), the asymptotic formula (5.1) becomes exact and the remainder $O(\hbar)$ in (5.2) is absent. Indeed, let us compare (5.1), (5.2) with formula (3.7) for the magnetic product $\underset{F}{\star}$. First, one can describe the symplectic groupoid $\mathcal{E}$ in our specific case. Let us set $\mathcal{E}=T^*\mathbb{R}^{2n}=\mathbb{R}^{2n}_x\oplus\mathbb{R}^{2n}_y$, and denote by (x,y) points in $\mathcal{E}$. The space $\mathbb{R}^{2n}_x$ is imbedded into $\mathcal{E}$ as a zero section $\{y=0\}$. The symplectic form on $\mathcal{E}$ is $dy\wedge dx$. The operators L,R (2.10) of left and right regular representation of algebra (1.3) generate two mappings

$$l:\mathcal{E}\to\mathbb{R}^{2n},\quad r:\mathcal{E}\to\mathbb{R}^{2n}. \tag{5.3}$$

Here $l=(l_q,l_p)$ and $r=(r_q,r_p)$ are just symbols of $L=(L_q,L_p)$ and $R=(R_q,R_p)$, i.e.,

$$L=l(x,-i\hbar\partial_x),\quad R=r(x,-i\hbar\partial_x).$$

In our case formula (2.10) reads

$$\begin{aligned} &l_q(x,y)=x_q-y_p/2, &&l_p(x,y)=x_p+y_q/2-A(l_q,r_q),\\ &r_q(x,y)=x_q+y_p/2, &&r_p(x,y)=x_p-y_q/2-A(r_q,l_q). \end{aligned} \tag{5.4}$$

Mappings (5.3) are Poisson and anti-Poisson, i.e., l preserves brackets, r changes the sign of brackets (recall that on $\mathbb{R}^{2n}$ we have the bracket (3.4), and the bracket on $\mathcal{E}$ corresponds to the symplectic form $dy\wedge dx$).

The groupoid structure on $\mathcal{E}$ is defined as follows: points $m_2,m_1\in\mathcal{E}$ are called *multiplicable* iff $r(m_2)=l(m_1)$; the product $m=m_2\circ m_1$, by definition, is a point in $\mathcal{E}$ such that $l(m)=l(m_2)$, $r(m)=r(m_1)$. The subspace $\mathbb{R}^{2n}_x\subset\mathcal{E}$ is the set of units of this groupoid, and mappings (5.3) are left and right reduction mappings:

$$l(m)=m\circ m^{-1},\qquad r(m)=m^{-1}\circ m.$$

The graph $\Lambda_*\subset\mathcal{E}\times\mathcal{E}\times\mathcal{E}$ of this groupoid multiplication consists of all multiplicable points and their products

$$\Lambda_*=\{(m,m_2,m_1)|m=m_2\circ m_1\}. \tag{5.5}$$

If on $\mathcal{E}\times\mathcal{E}\times\mathcal{E}$ we introduce the symplectic form $dy\wedge dx-dy_2\wedge dx_2-dy_1\wedge dx_1$, then the submanifold Λ_* is Lagrangian (see details in [29, 44]).

In the case $F=0$ (i.e., A=0) formulas (5.4) are interpreted as "middle point of chord" relations: $x=\frac{1}{2}(l+r)$, $y=JV$, where $V=l-r\in T\mathbb{R}^{2n}$. So, the groupoid structure on $\mathbb{R}^{4n}=T^*\mathbb{R}^{2n}$ is given by the triangle rule (see Fig. 7):

$$\begin{aligned} &m=m_2\circ m_1,\quad m=(x,y),\quad m_1=(x_1,y_1),\quad m_2=(x_2,y_2);\\ &y=y_1+y_2,\qquad x=x_1+\tfrac{1}{2}J^{-1}y_2=x_2-\tfrac{1}{2}J^{-1}y_1. \end{aligned} \tag{5.6}$$

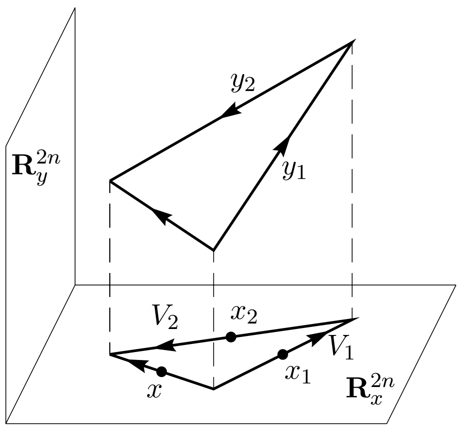

Figure 7.

For arbitrary given triple of points $x, x_2, x_1 \in \mathbb{R}^{2n}$ we uniquely construct the triangle for which these points are the middle points of its sides, and so reconstruct elements $m, m_2, m_1 \in \mathcal{E}$ such that $m = m_2 \circ m_1$. This means that the graph Λ_* is one-to-one projected onto $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ along the "vertical" y-polarization. Hence, the kernel $K_\hbar$ has the WKB form (5.2), and its phase is

$$S(x, x_2, x_1) = \int_{(0,0,0)}^{(x,x_2,x_1)} (y dx - y_2 dx_2 - y_1 dx_1). \tag{5.7}$$

Here y, y_2, y_1 are determined via x, x_2, x_1 following the triangle multiplication rule; the initial point $(0,0,0)$ corresponds to triple of elements $m_0 = m_0 \circ m_0$, where $m_0 = (0,0) \in \mathcal{E}$.

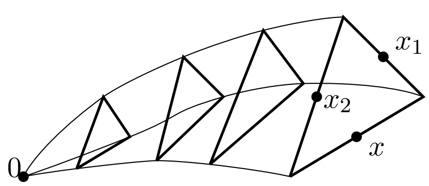

Figure 8.

The integral (5.7) is taken over an arbitrary path on Λ_* connecting the triple $(0,0,0)$ (i.e., the degenerate triangle) with the triple (x, x_2, x_1) (i.e. the given triangle). This path is actually a family of triangles in $\mathbb{R}^{2n}$; see Fig. 8. The phase (5.7) in the case $F = 0$ is just equal to the area of the final triangle

$$S(x, x_2, x_1) = \int_{\triangle(x,x_2,x_1)} \omega_0.$$

Now let us see what happens in the general magnetic case $F \neq 0$. From (5.4) it follows that the groupoid structure now is given by formulas

$$x = \tfrac{1}{2}(l + r) + V_F, \quad y = JV + Y_F, \quad \text{where} \quad V = l - r. \tag{5.8}$$

Here we have introduced notations

$$V_F = (0; A^s) \in \mathbb{R}^n_q \oplus \mathbb{R}^n_p, \quad Y_F = (A^a; 0) \in \mathbb{R}^n_{y_q} \oplus \mathbb{R}^n_{y_p},$$
$$A^s = \tfrac{1}{2}(A(l_q, r_q) + A(r_q, l_q)), \quad A^a = A(l_q, r_q) - A(r_q, l_q). \tag{5.9}$$

Comparing with the multiplication rule (5.6) in the case $F = 0$ we see that points x, x_2, x_1 are no longer the middle points of sides of the basic triangle and do not even belong to those sides. They are shifted in the p-direction by the vectors V_F.

Another difference from the case $F = 0$ is that the usual multiplication rule $y = y_1 + y_2$ (see (5.6)) fails to hold for inhomogeneous magnetic case. New "magnetic" rule is

$$y = y_1 + y_2 + \Big(\int_\triangle \nabla F;\ 0\Big), \tag{5.10}$$

where the vector-valued closed 2-form ∇F is defined by $\nabla F = \frac{1}{2}\nabla F_{jk} dq^k \wedge dq^j$, and $\triangle = \triangle(q, q_2, q_1)$ is the triangle in q-space with middle points (q, q_2, q_1), which are projections of $x, x_2, x_1 \in \mathbb{R}^{2n}$ onto $\mathbb{R}^n_q$. Note that

$$-\int_\triangle \nabla F = \oint_{\partial\triangle} F\, dq \ \sim \ \text{Lorentz momentum.}$$

The vector 2-form ∇F is a measure of *magnetic inhomogeneity*. So, this form controls the modification of the usual ($F = 0$) multiplication rule. The space $\mathbb{R}^{2n}_y$ (dual to $\mathbb{R}^{2n}_x$) now is not even a group, it is a pseudogroup over the Poisson manifold $\mathbb{R}^{2n}_x$ (see details in general case in [16], the word "pseudo"- reflects the fact that the product (5.10) in the y-space depends on x-coordinates as additional parameters, and the condition of "associativity" of the product (5.10) senses this dependence). The non-trivial part of the pseudogroup structure (5.10) is determined by the momentum of the membrane $\triangle$ in the magnetic field.

Formula (5.7) in the case $F \neq 0$ still represents the phase in the product (3.7) if we put there the corrected values $y = JV + Y_F$. But now the triangle no longer represents the groupoid multiplication rule, and from this point of view, it seems unnatural to keep this triangle in the formula for the phase.

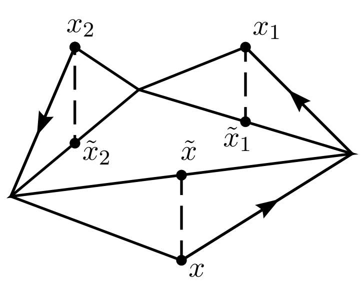

Figure 9.

Actually, we have another configuration (see Fig. 9) related to the triple (x, x_2, x_1). This configuration consists of triangle $\Sigma(\tilde{x}, \tilde{x}_2, \tilde{x}_1)$, where $\tilde{x} = x - V_F$, and also of three additional triangles directed "vertically" (parallel to p) with "top" vertices x, x_2, x_1. We call them *magnetic wings*.

A magnetic wing is characterized by a sequence of its vertices $[l, x, r]$ related to each other by (5.8) (see Fig. 10, left picture). The configuration of membrane with wings, as we see, exactly corresponds to the new groupoid multiplication rule for $F \neq 0$.

Figure 10.

Lemma 5

$$\int_{\Sigma(\tilde{x},\tilde{x}_2,\tilde{x}_1)} \omega_F = \int_{\Sigma(x,x_2,x_1)} \omega_F, \qquad \int_{\text{vertical wing}} \omega_F = 0.$$

Both statements of the Lemma follow from the orthogonality condition (2.2).

Denote by $\Sigma_F(x, x_2, x_1)$ the triangle $\Sigma(\tilde{x}, \tilde{x}_2, \tilde{x}_1)$ together with three magnetic wings described above. The boundary of this figure consists of six straight line segments. The figure itself looks like inflected hexagon. More generally, Σ_F could be any membrane in the phase space $\mathbb{R}^{2n}$ with that six segment boundary. In view of Lemma 5 we have

$$\int_{\Sigma(x,x_2,x_1)} \omega_F = \int_{\Sigma_F(x,x_2,x_1)} \omega_F.$$

Theorem 3 *The Weyl type magnetic product over $T^*\mathbb{R}^n$ is given by*

$$(f \underset{F}{\star} g)(x) = \frac{1}{(\pi\hbar)^{2n}} \int\int \exp\left\{\frac{i}{\hbar}\int_{\Sigma_F(x,x_2,x_1)} \omega_F\right\} f(x_2) g(x_1)\, dx_2\, dx_1, \tag{5.11}$$

where Σ_F is the wing membrane corresponding to the magnetic groupoid structure (5.8) *on $T^*(T^*\mathbb{R}^n)$.*

6 Ordering in quantization

Now we demonstrate that membrane wings introduced in previous section are actually very natural objects in the quantization framework. We show how different configurations of wings relate to different choices in the ordering problem. In particular, we'll see why the Weyl ordering choice (Weyl symmetrization) looks like an optimal choice.

Let us take a constant real $2n \times 2n$ matrix M (actually, a linear operator in the space tangent to $T^*\mathbb{R}^n$) which obeys the condition

$$M^T J + JM = 0, \tag{6.1}$$

where M^T is the transposed matrix. One can represent M by its $n \times n$ blocks as follows

$$M = \begin{pmatrix} N & K \\ S & -N^T \end{pmatrix}.$$

Now let us take an arbitrary function $f = f(q,p)$ (say, a polynomial) and determine the following general ordering of operators $\hat{q}, \hat{p}'$ (generators of the Heisenberg algebra):

$$\hat{f}^M \equiv f\Big(\tfrac{1}{2}N(\overset{4}{\hat{q}} - \overset{2}{\hat{q}}) + \tfrac{1}{2}(\overset{2}{\hat{q}} + \overset{4}{\hat{q}}) - \tfrac{1}{2}K(\overset{5}{\hat{p}'} - \overset{1}{\hat{p}'}),\ \overset{3}{\hat{p}'} - \tfrac{1}{2}S(\overset{4}{\hat{q}} - \overset{2}{\hat{q}}) - \tfrac{1}{2}N^T(\overset{5}{\hat{p}'} - \overset{1}{\hat{p}'})\Big). \tag{6.2}$$

We would like to obtain the product formula

$$\hat{f}^M \cdot \hat{g}^M = \hat{k}^M, \quad k = f \overset{M}{*} g. \tag{6.3}$$

To derive the product $\overset{M}{*}$ we first note that the M-ordering (6.2) is related to the Weyl ordering via

$$\hat{f}^M = (U^M f)(\hat{q}, \hat{p}'), \tag{6.4}$$

where

$$U^M = \exp\left\{\frac{i}{\hbar} S^M(\widehat{V})\right\}, \qquad \widehat{V} = (i\hbar\partial_p, i\hbar\partial_q),$$

and the function S^M is defined by the matrix M as follows

$$S^M(V) = \frac{1}{2}\langle JMV, V\rangle = \int_{\triangle^M(V)} \omega_0.$$

Here $\triangle^M(V)$ is the triangle, called the M-*wing*, generated by vector V and by vector MV applied at the middle point of V (see Fig. 10).

Using U^M we can calculate the product (6.3) by the formula

$$f \overset{M}{*} g \equiv U^{-M}(U^M f \star U^M g),$$

where $\star$ is the Weyl product (without magnetic correction, at first). Looking at exponential representation of the Weyl product, one concludes that

$$f \overset{M}{*} g = \exp\left\{\frac{i}{\hbar}\Phi^M(\widehat{V}_2, \widehat{V}_1)\right\} f(x_2) g(x_1)\Big|_{x_1 = x_2 = x},$$

where $\widehat{V} = (i\hbar\partial_p, i\hbar\partial_q)$. In this formula we use the notation

$$\Phi^M(V_2, V_1) = \int_{\Sigma(V_2,V_1)} \omega_0 + S^M(V_1) + S^M(V_2) - S^M(V_1 + V_2),$$

where $\Sigma(V_2, V_1)$ is the triangle generated by vectors V_1, V_2.

The phase function Φ^M can be written as the symplectic area of the *wing membrane* $\Sigma_x^M(V_2, V_1)$ generated by four triangles:

$$\Sigma_x^M(V_2, V_1) = \triangle(V_2, V_1) \bigcup \triangle^M(V_1) \bigcup \triangle^M(V_2) \bigcup \triangle^M(V_1 + V_2).$$

Its boundary consists of six line segments. For reasons of uniformity in notation the point x is shown; actually, the ω_0-area of $\Sigma^M_x(V_2, V_1)$ is independent of x.

The picture for Σ^M_x is the same as in Fig. 9, but wings now are M-wings as in Fig. 10 (right picture).

Theorem 4 *For an arbitrary $2n \times 2n$ matrix M, satisfying* (6.1), *there is a star-product over $\mathbb{R}^{2n}$ given by the wing membranes*:

$$(f \overset{M}{*} g)(x) = \exp\left\{\frac{i}{\hbar}\int_{\Sigma^M_x(\widehat{V}_2,\widehat{V}_1)} \omega_0\right\} f(x_2)g(x_1)\Big|_{x_1=x_2=x}. \tag{6.5}$$

This product corresponds to the M-ordering rule (6.2) *of non-commutative operators $\hat{q}, \hat{p}'$ (generators of the Heisenberg algebra), so that formula* (6.3) *holds.*

The family of orderings (6.2) was introduced and studied in detail in [30], where the following pseudodifferential formulas for M-product were obtained:

$$(f \overset{M}{\star} g)(x) = f\Big(\overset{2}{x} - i\hbar(\tfrac{1}{2} - M)J^{-1}\overset{1}{\partial_x}\Big)g(x) = g\Big(\overset{2}{x} + i\hbar(\tfrac{1}{2} + M)J^{-1}\overset{1}{\partial_x}\Big)f(x).$$

In the particular case

$$M = (\tfrac{1}{2} - \tau)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{6.6}$$

the ordering (6.2) simplifies to

$$\hat{f}^\tau = f\Big(\tau\overset{1}{\hat{q}} + (1-\tau)\overset{3}{\hat{q}}, \overset{2}{\hat{p}'}\Big). \tag{6.7}$$

The family of τ-wings corresponding to this family of orderings is represented in Fig. 11. The whole membrane Σ^M_x for the specific case $\tau = 0$ is shown in Fig. 12.

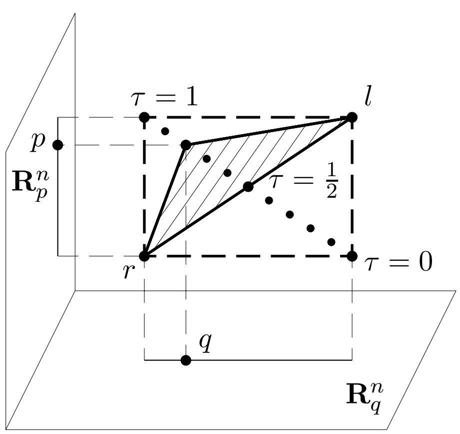

Figure 11.

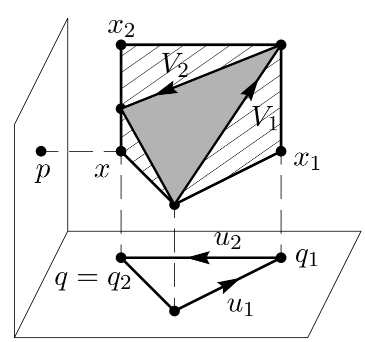

Figure 12.

The cases $\tau = 0$ and $\tau = 1$ are called the standard and anti-standard ordering choices (they correspond to push- and pull- groupoid structures on $T\mathbb{R}^n$, see Section 4), and the case $\tau = \frac{1}{2}$ is the Weyl ordering (corresponding to the saddle-groupoid structure on $T\mathbb{R}^n$). In the later case $M = 0$ and wings in membranes Σ^M are absent.

Matrix M is assumed to be real, but if we formally take $M = iJ/2$ (i.e., $N = 0$, $S = -i/2$, $K = i/2$), then the transform $U^M = \exp\{-\hbar(\partial_q^2 + \partial_p^2)/4\}$ relates Weyl ordering to the Wick normal ordering choice. In this case wings are pure imaginary (see Fig. 13), and the membrane Σ^M coincides with hexagon membrane introduced in [7, 8].

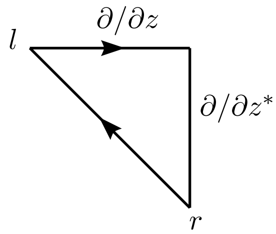

Figure 13.

Remark 2 In the general case, block N of the matrix M controls the position of the point $q \in \mathbb{R}^n$ with respect to the middle point of the vector u in the $T\mathbb{R}^n$-groupoid interpretation. The block S controls an additional $T\mathbb{R}^n$-groupoid cocycle (in u-coordinates) which appears in the convolution formula (see Section 4). In contrast to that, the block K in the matrix M generates the transformation of distributions over $T\mathbb{R}^n_q$ of the following type: $\exp\{-i\hbar K\partial_q \cdot \partial_q/2\}$. This is not a point transformation. Thus the product $\overset{M}{*}$ corresponding to matrix M with $K \neq 0$ can not be obtained from the groupoid convolution over $T\mathbb{R}^n$ by construction of Section 4. In particular, the Wick product is of such type.

Now let us consider the symplectic groupoid structure on the secondary phase space $T^*(T^*\mathbb{R}^n)$ corresponding to the $\overset{M}{*}$ product (6.3). In the same way as in Section 2 we calculate for the Heisenberg algebra (2.5) operators of left and right representations corresponding to the ordering choice (6.2). We know the transformation from (6.2) to the Weyl ordering; it is given by operator (6.4). Thus the left operators $L^M_q = q\overset{M}{*}$, $L^M_{p'} = p'\overset{M}{*}$ and right operators $R^M_q = \overset{M}{*}\, q$, $\mathbb{R}^M_{p'} = \overset{M}{*}\, p'$ are given by

$$L^M = U^{-M} \cdot L' \cdot U^M, \quad R^M = U^{-M} \cdot R' \cdot U^M, \tag{6.8}$$

where L', R' are determined in (2.8). We represent L^M, R^M via their symbols l, r:

$$L^M = l(x, -i\hbar\partial_x), \quad R^M = r(x, -i\hbar\partial_x),$$

and easily calculate l, r by (6.8) and by the definition of U^M. The result is

$$l(x, y) = x + (\tfrac{1}{2} - M)J^{-1}y, \quad r(x, y) = x - (\tfrac{1}{2} + M)J^{-1}y.$$

The inverse mapping $(l, r) \to (x, y)$ is given by

$$x = \tfrac{1}{2}(l + r) + MV, \quad V \equiv l - r = J^{-1}y. \tag{6.9}$$

For example, for the ordering cases (6.7), formulas (6.9) are represented in Fig. 11.

The corresponding symplectic groupoid structure on $T^*(T^*\mathbb{R}^n) = \mathbb{R}^{2n}_x \oplus \mathbb{R}^{2n}_y$ is given by the rule represented in Fig. 14. We observe that this structure is exactly given by the membrane with wings Σ^M which we described at the beginning of this section. What is new now is that we have identified the positions of points x, x_2, x_1 exactly as vertices of wings of the membrane Σ^M.

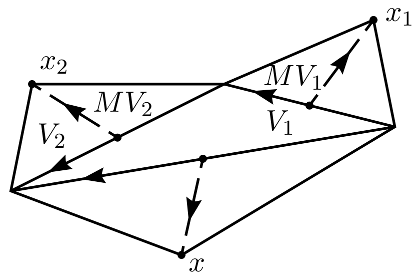

Figure 14.

Corollary 2 *Let* $\det(\frac{1}{2} - M) \neq 0$, *then the integral version of the product formula* (6.5) *reads*

$$(f \overset{M}{*} g)(x) = \frac{1}{(2\pi\hbar)^{2n}|\det(\frac{1}{2} - M)|} \int\int \exp\Big\{\frac{i}{\hbar}\int_{\Sigma^M(x,x_2,x_1)} \omega_0\Big\} f(x_2)g(x_1)dx_2dx_1, \tag{6.10}$$

where $\Sigma^M(x, x_2, x_1)$ *is the membrane with three* M*-wings having vertices* x, x_2, x_1.

Note that for the τ-ordering case (6.6) the denominator in formula (6.10) $\det(\frac{1}{2} - M) = \tau^n(1-\tau)^n$ is zero if $\tau = 0$ or $\tau = 1$. We see that the cases $\tau = 0$ (standard ordering) and $\tau = 1$ (anti-standard ordering) are special. In these cases the graph Λ_* of the groupoid multiplication (see Section 5) is not one-to-one projected onto the "configuration space" $\mathbb{R}^{2n}_x \times \mathbb{R}^{2n}_{x_2} \times \mathbb{R}^{2n}_{x_1}$, i.e., the graph is totally caustic and the integral kernel $K_\hbar(x, x_2, x_1)$ (5.1) is not of WKB-type (5.2). This is one of essential diffferences between representations (6.5) and (6.10) for the $*$-product.

Corollary 3 *For several multipliers there is the following product formula*

$$(f_N \overset{M}{*} \cdots \overset{M}{*} f_1)(x) = \exp\Big\{\frac{i}{\hbar}\int_{\Sigma^M_x(\widehat{V}_N,\dots,\widehat{V}_1)} \omega_0\Big\} f_N(x_N)\dots f_1(x_1)\Big|_{x_1=\cdots=x_N=x}, \tag{6.11}$$

where $\Sigma^M_x(V_N, \dots, V_1)$ *is the membrane with* $N+1$ *wings.*

These formulas are naturally generalized for products of differently quantized multipliers, i.e., for the case when each multiplier has its own ordering choice:

$$\begin{aligned} &\hat{f}^{M_N}_N \cdot \ldots \hat{f}^{M_1}_1 = \hat{k}^{M_{N+1}}, \\ &k(x) = \exp\Big\{\frac{i}{\hbar}\int_{\Sigma^{M_{N+1},\dots,M_1}_x(\widehat{V}_N,\dots,\widehat{V}_1)} \omega_0\Big\} f_N(x_N)\dots f_1(x_1)\Big|_{x_1=\cdots=x_N=x}. \end{aligned} \tag{6.12}$$

Here the membrane $\Sigma_x^{M_{N+1},\dots,M_1}$ is constructed by $N+1$ wings each of different configuration determined by different matrices $M_{N+1},\dots,M_1$.

In conclusion of this section we consider the magnetic case $F \neq 0$. For simplicity, we concentrate on the τ-ordering choice

$$\hat{f}^\tau = f(\tau \overset{1}{\hat{q}} + (1-\tau) \overset{3}{\hat{q}}, \overset{2}{\hat{p}}), \tag{6.13}$$

where $\hat{q}$, $\hat{p}$ satisfy the magnetic commutation relations (1.3).

We first transform the magnetic τ-ordering to the magnetic Weyl ordering:

$$\hat{f}^\tau = (U^\tau f)(\hat{q}, \hat{p}), \quad U^\tau = \exp\{(\tfrac{1}{2} - \tau)\hat{u}\partial_q\}, \tag{6.14}$$

where $\hat{u} = i\hbar\partial_p$. Then for product of two τ-ordered observables we have from (2.11):

$$\hat{f}^\tau \cdot \hat{g}^\tau = (U^\tau f)(\hat{q}, \hat{p}) \cdot (U^\tau g)(\hat{q}, \hat{p}) = (U^\tau f \underset{F}{\star} U^\tau g)(\hat{q}, \hat{p}),$$

where $\underset{F}{\star}$ is the magnetic product (3.2), (3.5), corresponding to the Weyl ordering. Transforming back the Weyl symbol to τ-symbol by the transform $(U^\tau)^{-1}$, we obtain the product formula

$$\hat{f}^\tau \cdot \hat{g}^\tau = \hat{k}^\tau,$$

where

$$k \equiv f \overset{\tau}{\underset{F}{\star}} g = (U^\tau)^{-1}(U^\tau f \underset{F}{\star} U^\tau g).$$

By formula (3.2) we calculate the new τ-magnetic product as follows

$$(f \overset{\tau}{\underset{F}{\star}} g)(q,p) = \tag{6.15}$$

$$\exp\{(\tau - \tfrac{1}{2})\hat{u}\partial_q\}\Big(\exp\{\tfrac{i}{\hbar}\phi(q, \hat{u}_2, \hat{u}_1) + \tfrac{1}{2}(\hat{u}_1\partial_{q_2} - \hat{u}_2\partial_{q_1})\}$$

$$\times \exp\{(\tfrac{1}{2} - \tau)(\hat{u}_1\partial_{q_1} + \hat{u}_2\partial_{q_2})\} f(q_2, p_2) g(q_1, p_1)\Big)\Big|_{q_1=q_2=q,\, p_1=p_2=p}$$

$$= \exp\Big\{\tfrac{i}{\hbar}\phi\Big(q + (\tau - \tfrac{1}{2})(\hat{u}_1 + \hat{u}_2), \hat{u}_2, \hat{u}_1\Big) + \tfrac{1}{2}(\hat{u}_1\partial_{q_2} - \hat{u}_2\partial_{q_1})$$

$$+(\tau - \tfrac{1}{2})\Big((\hat{u}_1 + \hat{u}_2)(\partial_{q_1} + \partial_{q_2}) - \hat{u}_1\partial_{q_1} - \hat{u}_2\partial_{q_2}\Big)\Big\} f(q_2, p_2) g(q_1, p_1)\Big|_{q_1=q_2=q,\, p_1=p_2=p}.$$

Note that the flux $\phi(q, u_2, u_1)$ was defined by (3.3) via the triangle $\triangle_q(u_2, u_1)$ in $\mathbb{R}^n$ with the middle point q of one of sides. Now we see that the position of the middle point $\tilde{q}$ differs from q by an additional vector $(\tau - 1/2)(u_1 + u_2)$. So,

$$q = \tilde{q} + (\tfrac{1}{2} - \tau)(u_1 + u_2) = \tilde{q} + (MV)_q,$$

where M is matrix (6.6) corresponding to τ-ordering, and $V = V_1 + V_2$ is a vector in $\mathbb{R}^{2n}$ whose q-component is $u = u_1 + u_2$.

The total phase in (6.14) is equal to the magnetic area of the membrane in $\mathbb{R}^{2n}$ constructed by the triangle with sides V_1, V_2, V_1+V_2 and by three additional wings generated by matrix M of special type (6.6). The configuration of the wings in this special case is shown in Fig. 11.

The position of the vertex of the wing is $x^\tau = \tilde{x}+MV$, where $\tilde{x} = (l+r)/2$. But actually this position should be changed to match the symplectic groupoid multiplication rule. To find this rule we have to calculate the operators of left and right regular representation in our τ-case. In view of (6.14) these operators are given by formulas

$$L^\tau = (U^\tau)^{-1} \cdot L \cdot U^\tau, \quad R^\tau = (U^\tau)^{-1} \cdot R \cdot U^\tau,$$

where L, R are operators (2.10), corresponding to the magnetic Weyl ordering. From (6.14) and (2.10) we obtain

$$\begin{array}{llll} L^\tau_q & = q + \tau i\hbar\partial_p, & R^\tau_q & = q - (1-\tau)i\hbar\partial_p, \\ L^\tau_p & = p - (1-\tau)i\hbar\partial_q - A(L^\tau_q, R^\tau_q), & R^\tau_p & = p + \tau i\hbar\partial_q - A(R^\tau_q, L^\tau_q). \end{array}$$

So, if we represent these operators by symbols $L^\tau = l(x, -i\hbar\partial_x)$, $R^\tau = r(x, -i\hbar\partial_x)$, then two groupoid mappings appear

$$l : T^*\mathbb{R}^{2n} \to \mathbb{R}^{2n}, \qquad r : T^*\mathbb{R}^{2n} \to \mathbb{R}^{2n},$$

where

$$\begin{aligned} l(x,y) &= x + (\tfrac{1}{2} - M)J^{-1}y - \big(0; A(l_q, r_q)\big), \\ r(x,y) &= x - (\tfrac{1}{2} + M)J^{-1}y - \big(0; A(r_q, l_q)\big). \end{aligned} \tag{6.16}$$

From (6.16) we reconstruct x and y via l, r

$$x = \tfrac{1}{2}(l+r) + V_F + MV, \quad y = JV + Y_F, \tag{6.17}$$

where $V = l - r$, and vectors V_F, Y_F are given by the same formulas as in (5.9) (i.e., the same as in the case $\tau = 1/2$ or $M = 0$).

Relations (6.17) determine the final configuration of the *magnetic τ-wing*. Position of the vertex x of this magnetic wing is shifted by vector V_F with respect to position x^τ.

Lemma 6 *The magnetic area of the magnetic τ-wing with the vertex x is equal to the magnetic area of the wing with vertex x^τ.*

In view of this lemma the phase (6.15) can be represented by the area of a membrane $\Sigma^\tau_x(V_2, V_1)$ constructed by the triangle of vectors V_1, V_2, V_1+V_2, and by three magnetic τ-wings over each of these vectors. The point x is the vertex of the magnetic τ-wing over V_1+V_2.

Theorem 5 *The magnetic star-product corresponding to τ-ordering* (6.13) *of noncommutative coordinates $\hat{q}, \hat{p}$ is given by formula*

$$(f \overset{\tau}{\underset{F}{\star}} g)(x) = \exp\Big\{\frac{i}{\hbar}\int_{\Sigma^\tau_x(\widehat{V}_2,\widehat{V}_1)} \omega_F\Big\} f(x_2)g(x_1)\Big|_{x_1=x_2=x},$$

where Σ^τ_x is a membrane with magnetic τ-wings.

The immediate corollaries from this statement are formulas for several multipliers and also the integral formula for $\overset{\tau}{\underset{F}{\star}}$ via the magnetic area of membranes $\Sigma^\tau(x, x_2, x_1)$ supplied with magnetic τ-wings. These corollaries are formulated by the same way as (6.10)–(6.12), but with magnetic form ω_F in the exponent.

7 Dynamics via membrane area

Quantum dynamics of a charged (spinless) particle in electromagnetic field can be described

– in the nonrelativistic case by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \Big[\frac{1}{2m}\Big(-i\hbar\frac{\partial}{\partial q} - \frac{e}{c}\mathcal{A}\Big)^2 + ea\Big]\psi, \tag{7.1}$$

– in the relativistic case by the Klein–Gordon equation

$$\Big(i\hbar\frac{\partial}{\partial t} - ea\Big)^2\psi = c^2\Big[\Big(-i\hbar\frac{\partial}{\partial q} - \frac{e}{c}\mathcal{A}\Big)^2 + m^2c^2\Big]\psi, \tag{7.2}$$

where $\mathcal{A}$ and a are magnetic and electric potentials of the field, and metric is assumed to be Euclidean.

First, we consider the pure magnetic time-independent situation $a = 0$, $\mathcal{A} = \mathcal{A}(q)$. Let us introduce operators $\hat{p} = -i\hbar\partial/\partial q - \frac{e}{c}\mathcal{A}(q)$. Then the dynamical equations can be reduced to studying the evolution operator $\exp\{-itH(\hat{p})/\hbar\}$, where $H(p) = p^2/2m$ in the nonrelativistic case, and $H(p) = \pm c\sqrt{p^2 + m^2c^2}$ in the relativistic case.

More generally, in a presence of an additional non-Euclidean metric (gravitational field) the Hamiltonian will depend on the q-coordinate as well: $H = H(q, p)$. Say, $H \simeq g^{jk}(q)p_jp_k/2m$ in the nonrelativistic case.

So, the general problem is to study the operator

$$U^t = \exp\Big\{-\frac{it}{\hbar}\widehat{H}\Big\}, \qquad \widehat{H} = H(\hat{q}, \hat{p}). \tag{7.3}$$

In particular, we are interested in its asymptotic behavior as $\hbar \to 0$.

Note that the symbol H and the permutation relations between quantum coordinates $\hat{q}, \hat{p}$ (1.3) are independent of the gauge choice of the magnetic potential $\mathcal{A}$. Thus the semiclassical approximation for U^t as $\hbar \to 0$, written in terms of the phase space symbol $H(q, p)$ and the phase space noncommutative structure (1.3), is automatically gauge invariant.

Let us represent the operator U^t in the Weyl form

$$U^t = \mathcal{U}^t(\hat{q}, \hat{p}),$$

then for symbol $\mathcal{U}^t$ one obtains the following equations

$$i\hbar\frac{\partial\mathcal{U}^t}{\partial t} = H \underset{F}{\star} \mathcal{U}^t, \qquad \mathcal{U}^0 = 1. \tag{7.4}$$

Using Theorem 1 we transform (7.4) to a pseudodifferential form:

$$i\hbar \frac{\partial \mathcal{U}^t(x)}{\partial t} = \mathcal{H}_\hbar(x, -i\hbar\partial_x)\,\mathcal{U}^t(x), \qquad \mathcal{U}^0 = 1. \tag{7.5}$$

Here $\mathcal{H}_\hbar(x, y)$ is the Weyl symbol of the operator $H(L_q, L_p)$, and L_q, L_p are given by (2.10). Obviously,

$$\mathcal{H}_\hbar = \mathcal{H}_0 + O(\hbar^2), \qquad \mathcal{H}_0 \equiv H(l), \tag{7.6}$$

where l is the Weyl symbol of L, i.e., $L = l(x, -i\hbar\partial_x)$. The explicit formulas for $l = (l_q, l_p)$ are found in (5.4).

In view of (7.6) the principal term of the semiclassical solution of the Cauchy problem (7.5) is determined by the Hamilton function $\mathcal{H}_0$. For small enough time interval $t \in [0, T]$ the approximate solution has the simplest WKB-form

$$\mathcal{U}^t(x) = \exp\Big\{\frac{i}{\hbar} S(t, x)\Big\} u^t(x) + O(\hbar), \tag{7.7}$$

where the phase S is the solution of the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \mathcal{H}_0\Big(x, \frac{\partial S}{\partial x}\Big) = 0, \qquad S\Big|_{t=0} = 0, \tag{7.8}$$

and the non-oscillatory amplitude u^t is the solution of the "transport" equation

$$\frac{\partial u^t}{\partial t} + \frac{\partial}{\partial x}\bigg(\frac{\partial \mathcal{H}_0}{\partial y}\Big(x, \frac{\partial S}{\partial x}\Big) u^t\bigg) = 0, \qquad u^0 = 1. \tag{7.9}$$

In order to solve (7.8), (7.9) one has to consider the Hamiltonian system

$$\dot{X} = \frac{\partial \mathcal{H}_0}{\partial Y}, \qquad \dot{Y} = -\frac{\partial \mathcal{H}_0}{\partial X}, \qquad X\Big|_{t=0} = x^0, \qquad Y\Big|_{t=0} = 0, \tag{7.10}$$

and determine

$$S = \int_0^t (Y\dot{X} - \mathcal{H}_0)\, d\tilde{t}, \qquad u^t = \Big(\frac{\mathcal{D}X}{\mathcal{D}x^0}\Big)^{-1/2}, \tag{7.11}$$

where $x^0 = (q^0, p^0)$ is taken from the equation

$$X(x^0, t) = x.$$

The time interval $t \in [0, T]$ over which the Jacobian $\mathcal{D}X/\mathcal{D}x^0 \geq \delta > 0$ (for any $x^0 \in \mathbb{R}^{2n}$) is exactly the interval where the solution $\mathcal{U}^t$ can be represented in the WKB form (7.7).

Lemma 7 *Trajectories of* (7.10) *are given by formulas*

$$\begin{aligned} X(x^0, t) &= \tfrac{1}{2}(\gamma^t(x^0) + x^0) + v^t(x^0), \\ Y(x^0, t) &= J(\gamma^t(x^0) - x^0) + y^t(x^0), \end{aligned} \tag{7.12}$$

where

$$v^t = \Big(0; \tfrac{1}{2}\big(A(\gamma_q^t, q^0) + A(q^0, \gamma_q^t)\big)\Big), \qquad y^t = \Big(A(\gamma_q^t, q^0) - A(q^0, \gamma_q^t); 0\Big),$$

and $\gamma^t = (\gamma_q^t, \gamma_p^t)$ *is the trajectory of the Hamiltonian system corresponding to the function* H *and the magnetic Poisson bracket* (3.4)

$$\dot\gamma_q = \frac{\partial H}{\partial p}(\gamma_q, \gamma_p), \qquad \dot\gamma_p = -\frac{\partial H}{\partial q}(\gamma_q, \gamma_p) - F(\gamma_q)\frac{\partial H}{\partial p}(\gamma_q, \gamma_p), \tag{7.13}$$

$$\gamma\Big|_{t=0} = x^0 = (q^0, p^0).$$

The proof of this lemma follows from (5.8) and from the fact that $\mathcal{H}_0 = H(l)$, and so components of the mapping $r : \mathbb{R}_x^{2n} \oplus \mathbb{R}_y^{2n} \to \mathbb{R}^{2n}$ (5.4) are integrals of motion for system (7.10), i.e., $r(X, Y) = r(x^0, 0) = x^0$ is constant in time.

After substitution of (7.12) into (7.11) one obtains the following result.

Theorem 6 *Let* $\hat q, \hat p$ *satisfy commutation relations* (1.3) *with magnetic tensor* F_{kj}, *and* $\widehat{H} = H(\hat q, \hat p)$. *Then for small enough time* t *the semiclassical approximation for the magnetic Weyl symbol of the evolution operator* $\exp\{-\frac{it}{\hbar}\widehat{H}\} = \mathcal{U}^t(\hat q, \hat p)$ *is given by the formula*

$$\mathcal{U}^t = \mathcal{J}^{-1/2} \exp\Big\{\frac{i}{\hbar}\int_\Sigma \omega_F - \frac{it}{\hbar}H\Big\} + O(\hbar). \tag{7.14}$$

Here the membrane $\Sigma = \Sigma_x^t$ *(see Fig.* 15, *left picture) is constructed from the piece of the Hamilton trajectory* (7.13), *which connects points* x^0 *and* $\gamma^t(x^0)$, *and from the magnetic wing with vertices* $[\gamma^t(x^0), x, x^0]$, *where*

$$x = \tfrac{1}{2}\big(\gamma^t(x^0) + x^0\big) + v^t(x^0). \tag{7.15}$$

Here

$$v^t(x^0) = \Big(0\ ; \tfrac{1}{2}\big(A(\gamma^t(x^0)_q, q^0) + A(q^0, \gamma^t(x^0)_q)\big)\Big), \qquad x^0 \equiv (q^0, p^0),$$

and A *is the Valatin primitive* (2.1). *The Jacobian* $\mathcal{J}$ *in* (7.14) *is given by*

$$\mathcal{J}(x^0, t) = \det[\tfrac{1}{2}(I + d\gamma^t(x^0)) + dv^t(x^0)], \qquad \mathcal{J} \geq \delta > 0 \text{ for } t \in [0, T],$$

the Hamilton function H *in* (7.14) *is evaluated on the trajectory, i.e.,* $H = H(x^0)$, *and* x^0 *is assumed to be expressed in terms of* x, t *via the equation* (7.15).

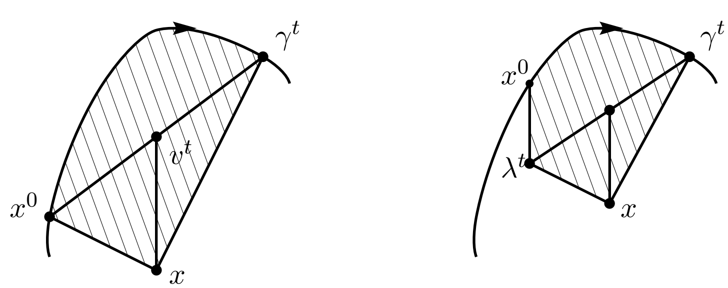

Figure 15.

Remark 3 The group property of the family of symbols $\mathcal{U}^t$ over $\mathbb{R}^{2n}$ reads

$$\mathcal{U}^{t_2} \underset{F}{\star} \mathcal{U}^{t_1} = \mathcal{U}^{t_2+t_1}. \tag{7.16}$$

In terms of WKB-phase functions (7.14), the identity (7.16) requires that

$$\int_{\Sigma^{t_2}_{x_2}} \omega_F + \int_{\Sigma^{t_1}_{x_1}} \omega_F + \int_{\Sigma_F(x, x_2, x_1)} \omega_F = \int_{\Sigma^{t_2+t_1}_{x}} \omega_F \tag{7.17}$$

(see Fig. 16), where $\Sigma_F(x, x_2, x_1)$ is the hexagon membrane with magnetic wings, defined at the end of Section 5. Note that in the case $F = 0$ formula (7.17) coincides with the phase addition rule obtained by Marinov [4]. In that particular case the magnetic "anomaly" v^t in (7.15) is absent and the magnetic wings of the membranes disappear.

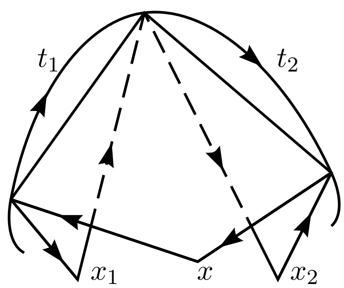

Figure 16.

Also note that the Hamilton function H could be time-dependent. In this case the first membrane phase factor in formula (7.14) is the same, but the second phase factor becomes $\exp\{-\frac{i}{\hbar}\int_0^t H(\gamma^{\tilde{t}}(x^0), \tilde{t})\, d\tilde{t}\}$; the trajectory γ^t is now the solution of system (7.13) with time-dependent Hamiltonian H.

Remark 4 One can use not only Weyl but any other ordering choice to represent the evolution operator as a function in coordinates $\hat{q}, \hat{p}$. Then formula (7.14) still holds with membrane Σ constructed by wings corresponding to the given ordering choice (see Section 6); equation (7.15) and the Jacobian $\mathcal{J}$ are changed following (6.17). Moreover, in [30] it was proved that using and combining different orderings it is possible to avoid the difficulty with time limitations $t \in [0, T]$ where the WKB-approximation works.

For the Wick ordering choice the wings are pure imaginary (see Fig. 13) and the membrane representation (7.14) coincides with those obtained in [9]. In this case the Jacobian $\mathcal{J}$ is never zero and representation (7.14) is global in t.

In conclusion of this section we apply formula (3.6) to derive the symbol $\mathcal{U}^t$ not asymptotically but in an exact continual form. Namely, one can use the approximation $\mathcal{U}^{t/N} = \exp\big\{ -\frac{it}{\hbar N} H\big\} + O(N^{-2})$ and obtain the Trotter type formula

$$\mathcal{U}^t = \lim_{N\to\infty} \exp\Big\{ -\frac{it}{\hbar N} H\Big\} \underset{F}{\star} \dots \underset{F}{\star} \exp\Big\{ -\frac{it}{\hbar N} H\Big\}$$

(N multipliers). Applying (3.6), one derives

$$\mathcal{U}^t(x) = \lim_{N\to\infty} \exp\Big\{\frac{i}{\hbar}\int_{\Sigma_x(\widehat{V}_N,\dots,\widehat{V}_1)} \omega_F\Big\} \exp\Big\{-\frac{it}{\hbar N}\sum_{j=1}^{N} H(x_j)\Big\}\Big|_{x_1=\cdots=x_N=x}. \tag{7.18}$$

Now the question is how to represent this formula in a continual form.

Note that each vector field v on $\mathbb{R}^{2n}$ and any point $x \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$ determines a membrane $\Sigma_x^t(v) \subset \mathbb{R}^{2n}$ whose boundary is constructed from a piece $\{\Gamma^\mu \mid 0 \le \mu \le t\}$ of the trajectory of the field v in $\mathbb{R}^{2n}$ and from the magnetic wing with vertices $[\Gamma^t, x, \Gamma^0]$ (or, the magnetic τ-wing if one wants to use the general τ-ordering). The integral over the membrane in (7.18) is an approximation of the integral over $\Sigma_x^t(v)$ with a convenient choice of v.

Theorem 7 *The following continual formula for the symbol of the evolution operator* (7.3) *holds*:

$$\mathcal{U}^t(x) = \exp\Big\{\frac{i}{\hbar}\int_{\Sigma_x^t(\widehat{v})} \omega_F\Big\} \exp\Big\{-\frac{i}{\hbar}\int_0^t H\big(x(\mu)\big)\,d\mu\Big\}\Big|_{x(\mu)\equiv x}. \tag{7.19}$$

Here $\big\{x(\mu) = \big(q(\mu), p(\mu)\big) \mid 0 \le \mu \le t\big\}$ *are continuous paths in* $\mathbb{R}^{2n}$ *and* $\widehat{v} = \big(i\hbar\delta/\delta p(\mu),$ $i\hbar\delta/\delta q(\mu)\big)$ *is the variational derivative operator acting on the path functional.*

Formula (7.19) is dual to the Feynman path-integral formula [45, 46, 47, 48, 21]. The difference between (7.19) and the path integral is the same as between (3.5) and (3.7). The known Wick and Hori formulas [49, 50] for the symbol of the evolution operator (see also generalizations in [51]) are structurally close to (7.19), but use a different first exponential factor. The membrane exponential factor in (7.19) clearly demonstrates the influence of the magnetic form ω_F to the quantum dynamics.

8 Electromagnetic fields and space-time membranes

Let us now consider general time-dependent case, i.e., $\mathcal{A} = \mathcal{A}(t, q)$, $a = a(t, q)$ in (7.1), (7.2). We again study the Cauchy problem for the Schrödinger or Klein–Gordon equations.

As a first step one can remove the electric potential a from equations by introducing the new wave function $\psi \exp\{\frac{ie}{\hbar}\int_0^t a\,dt\}$. After such a transform the magnetic potential $\mathcal{A}$ is replaced by $\mathcal{A} + c\int_0^t (\partial a/\partial q)\,dt$, but the electromagnetic tensor F_{jk} (1.2b) remains unchanged. So, without loss of generality one can assume that

$$a \equiv 0, \qquad E = -c^{-1}\partial\mathcal{A}/\partial t, \qquad B = \operatorname{curl}\mathcal{A}.$$

The quantum dynamical equations have the following form:
– in the nonrelativistic case

$$i\hbar\frac{\partial\psi}{\partial t} = H(\hat{q}, \hat{p}(t))\psi, \qquad \text{where} \quad H \simeq \frac{1}{2m} g^{jk}(q) p_j p_k; \tag{8.1a}$$

– in the relativistic case

$$\hbar^2 \frac{\partial^2 \psi}{\partial t^2} + H^2(\hat{q}, \hat{p}(t))\psi = 0, \qquad \text{where} \quad H^2 \simeq c^2(g^{jk}(q)p_j p_k + m^2 c^2). \tag{8.1b}$$

In the latter case the metric g^{jk} is assumed to be non-negative definite; the symbols $\simeq$ in (8.1 a,b) mean that some terms of order $\hbar$, $\hbar^2$ could be added to the Hamilton function [34, 52, 37, 20, 25, 53].

In equations (8.1) for each fixed time t the operators $\hat{p}(t) = -i\hbar\partial/\partial q - \frac{e}{c}\mathcal{A}(t, q)$ and $\hat{q} = q$ satisfy relations (1.3) with time-dependent tensor

$$F_{jk}(t, q) = \frac{e}{c}\epsilon_{kjl} B^l(t, q), \qquad q \in \mathbb{R}^3, \quad j, k = 1, 2, 3.$$

The time derivative of the operators $\hat{p}(t)$ in (8.1) is the following:

$$\frac{d}{dt}\hat{p}_j(t) = eE_j(t, q), \qquad j = 1, 2, 3. \tag{8.2}$$

So we see that the electric field is responsible for “dynamical evolution” of the quantum magnetic algebra (1.3).

Let us introduce two-point electric potential

$$\beta(t, q, q') \equiv \int_q^{q'} E(t, \tilde{q})\, d\tilde{q} \tag{8.3}$$

(the integral is taken along the straight line segment), and also the two-point magnetic potential

$$\alpha(t, q, q') \equiv \frac{1}{|q - q'|} \int_{q'}^{q} |\tilde{q} - q'| B(t, \tilde{q})\, d\tilde{q}. \tag{8.4}$$

We stress that these potentials are different from those used by Valatin [17] in the time dependent case, since in our present definitions there is no integration over the time variable. Time and space are separated because we study the Cauchy problem in time.

Lemma 8 *The relation holds*:

$$-\frac{\partial \beta}{\partial q} - \frac{1}{c}\frac{\partial \alpha}{\partial t} = E(t, q).$$

Now from (8.2) and from composition formulas (2.11) we obtain the following statement.

Proposition 6 (i) *The time derivative of any Weyl function in quantum coordinates $\hat{q}$, $\hat{p}(t)$ is given by*

$$-i\hbar \frac{d}{dt} f\big(\hat{q}, \hat{p}(t)\big) = f^e\big(\hat{q}, \hat{p}(t), t\big),$$

where

$$f^e(q, p, t) = e\beta(t, L_q, R_q) f(q, p),$$

and $L_q = q + \frac{1}{2}i\hbar\partial_p$, $R_q = q - \frac{1}{2}i\hbar\partial_p$ *are operators of the regular representation* (2.10).

(ii) *The composition of two Weyl functions is given by*

$$[f_2(\hat{q}, \hat{p}(t))] \cdot [f_1(\hat{q}, \hat{p}(t))] = k(\hat{q}, \hat{p}(t)), \qquad k = f_2(L_q, L_p(t)) f_1$$

where $L_p(t) = p - \frac{1}{2}i\hbar\partial_q - \frac{e}{c}\alpha(t, L_q, R_q)$.

The solution of the evolution problem (8.1a) has the general form

$$\psi(t,q) = \mathcal{U}^t(\hat{q}, \hat{p}(t))\Big(\psi\Big|_{t=0}\Big). \tag{8.5}$$

In view of Proposition 6 equations for symbol $\mathcal{U}^t$ are the following

$$\Big[-i\hbar\frac{\partial}{\partial t} + e\beta(t, L_q, R_q) + H(L_q, L_p(t))\Big]\mathcal{U}^t(x) = 0, \qquad \mathcal{U}^0 = 1. \tag{8.6}$$

The operator acting on $\mathcal{U}^t$ can be represented (as in (7.5)) via a symbol $\mathcal{H}_\hbar$ over $\mathbb{R}_t \times \mathbb{R}^6_x \times \mathbb{R}^6_y$. In the same way as in (7.6) we have

$$\mathcal{H}_\hbar = \mathcal{H}_0 + O(\hbar^2), \qquad \mathcal{H}_0(t,x,y) = e\beta\big(t, l_q(x,y), r_q(x,y)\big) + H\big(l_q(x,y), l_p(t,x,y)\big), \tag{8.7}$$

where $x = (q,p)$, $y = (y_q, y_p)$, and

$$l_q = q - \tfrac{1}{2}y_p, \qquad r_q = q + \tfrac{1}{2}y_p, \qquad l_p = p + \tfrac{1}{2}y_q - \tfrac{e}{c}\alpha(t, l_q, r_q).$$

(Of course, here we just re-state identities (5.4) in a new notation.) As in Section 7, the WKB-solution of (8.6) has the form (7.7), (7.11), where (X,Y) is now the trajectory of the Hamiltonian system

$$\dot{X} = \frac{\partial \mathcal{H}_0}{\partial y}(t,X,Y), \qquad \dot{Y} = -\frac{\partial \mathcal{H}_0}{\partial x}(t,X,Y), \qquad X\Big|_{t=0} = x^0, \qquad Y\Big|_{t=0} = 0. \tag{8.8}$$

The solution of problem (8.1b) with additional Cauchy data $\partial\psi/\partial t\big|_{t=0} = 0$ can also be constructed in the form (8.5), where $\mathcal{U}^t$ satisfies the equations

$$\Big[i\hbar\frac{\partial}{\partial t} - e\beta(t, L_q, R_q)\Big]^2 \mathcal{U}^t = H^2(L_q, L_p(t))\mathcal{U}^t, \qquad \mathcal{U}^0 = 1, \quad \frac{\partial}{\partial t}\mathcal{U}^t\Big|_{t=0} = 0.$$

The WKB-approximation has the form

$$\mathcal{U}^t = \frac{1}{2}\sum_{\pm} \exp\Big\{\frac{i}{\hbar}S_\pm\Big\}u^t_\pm + O(\hbar), \tag{8.9}$$

where the phases $S_\pm$ and amplitudes $u^t_\pm$ correspond (by formulas (7.11)) to the Hamilton function $\mathcal{H}_0$ of type (8.7) with $\pm$ signs in the definition of H. The Hamiltonian system (8.8) again plays the basic role.

The difference in Hamiltonian system (8.8) from the earlier (7.10) is that function $\mathcal{H}_0$ in (8.8) now depends on l_q, l_p and on r_q as well. So, r_p is not an integral of motion for (8.8).

Thus instead of dynamical system (7.13) we get now two systems: one for $\gamma = l(t, X, Y)$ and another for $\lambda = r(t, X, Y)$. They are the following:

$$\begin{aligned} \dot{\gamma}_q &= \frac{\partial H}{\partial p}(\gamma_q, \gamma_p), \\ \dot{\gamma}_p &= -\frac{\partial H}{\partial q}(\gamma_q, \gamma_p) - \frac{e}{c} B(t, \gamma_q) \times \dot{\gamma}_q + eE(t, \gamma_q), \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} \dot{\lambda}_q &= 0, \\ \dot{\lambda}_p &= eE(t, \lambda_q) \end{aligned} \tag{8.11}$$

with one and the same initial condition $\gamma\Big|_{t=0} = \lambda\Big|_{t=0} = x^0$.

The function $H(q, p)$ has the following form: $H(q, p) = g^{jk}(q) p_j p_k / 2m$ in the nonrelativistic case and $H(q, p) = \pm c\sqrt{g^{jk}(q) p_j p_k + m^2 c^2}$ in the relativistic case.

Note that (8.10) is the standard dynamical system for charged massive particle in the electromagnetic field. The additional system (8.11) can be interpreted as the dynamical system for a particle of charge e and mass $m = \infty$. The appearance of this additional "virtual particle" is due to the presence of the electric field E.

The phase S of the WKB-solution is given by (7.11); hence,

$$S(t, x) = \int_0^t Y\dot{X}\, d\tilde{t} - e\int_0^t \beta(\tilde{t}, \gamma_q^{\tilde{t}}, \lambda_q^{\tilde{t}})\, d\tilde{t} - \int_0^t H(\gamma^{\tilde{t}})\, d\tilde{t}. \tag{8.12}$$

Lemma 9 *The following identity holds*:

$$\int_0^t Y\dot{X}\, d\tilde{t} = \int_{\Sigma_x^t} \omega_F = \int_{\Sigma_x^t} \omega_0 + \frac{e}{c}\int_{\widetilde{\Sigma}_q^t} B(t, \tilde{q})\, d\tilde{q} \wedge d\tilde{q}, \tag{8.13}$$

where the membrane $\Sigma_x^t \subset \mathbb{R}^6 = T^*\mathbb{R}^3$ is constructed from the two trajectories $\gamma = \gamma^t$ (8.10) and $\lambda = \lambda^t$ (8.11) and from the magnetic wing with vertices $[\gamma^t, x, \lambda^t]$, where $x = (q, p)$ (see Fig. 15, right picture). The projection of Σ_x^t onto $\mathbb{R}^3$ is the membrane $\widetilde{\Sigma}_q^t$ constructed by a piece of the trajectory $\{\gamma_q^{\tilde{t}} \,|\, 0 \le \tilde{t} \le t\}$ and by the chord $[q^0, \gamma_q^t]$ with middle point q.

This is the membrane area interpretation of the first term in (8.12). The second term in view of definition (8.3) can also be written as two-dimensional area, but in extended space-time:

$$-e\int_0^t \beta\, d\tilde{t} = e\int_0^t d\tilde{t} \int_{q^0}^{\gamma_q^{\tilde{t}}} E(\tilde{t}, \tilde{q})\, d\tilde{q} = e\int_{\widetilde{\Sigma}_{t,q^0}} E\, dq \wedge dt. \tag{8.14}$$

Here $\widetilde{\Sigma}_{t,q^0}$ is a membrane in $\mathbb{R}_t \times \mathbb{R}^3_q$ whose boundary consists of the trajectory (the world line) $\{(\tilde{t}, \gamma^{\tilde{t}}) \,|\, 0 \le \tilde{t} \le t\}$ + the chord $[\gamma^t, q^0]$ + the straight time-segment $\{(\tilde{t}, q^0) \,|\, 0 \le \tilde{t} \le t\}$.

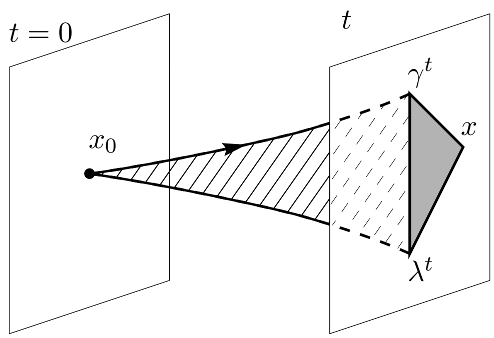

Figure 17.

Now one can combine (8.13), (8.14) and apply the Stokes theorem to transform the integration area to be of the most elegant geometry. Let us denote by $\Sigma_{t,x}$ the membrane in $\mathbb{R}_t \times \mathbb{R}^6_x$ whose boundary is constructed by the world line of the given particle $\{(\tilde{t}, \gamma^{\tilde{t}}) \,|\, 0 \le \tilde{t} \le t\}$, the world line of the virtual infinitely heavy particle $\{(\tilde{t}, \lambda^{\tilde{t}}) \,|\, 0 \le \tilde{t} \le t\}$, and also by the magnetic wing with vertices $[\lambda^t, x, \gamma^t]$ (see Fig. 17). We refer to $\Sigma_{t,x}$ as a *dynamical membrane.*

Proposition 7 *The WKB-phase of symbol* $\mathcal{U}^t$ *in* (8.5) *can be represented as*

$$S(t,x) = \int_{\Sigma_{t,x}} (\omega_0 + F) - \int_0^t H(\gamma^{\tilde{t}})\, d\tilde{t}, \tag{8.15}$$

where $\omega_0 = \frac{1}{2} J dx \wedge dx$*, the 2-form* F *is given by* (1.2b)*,* $\Sigma_{t,x}$ *is the dynamical membrane in* $\mathbb{R}^7 = \mathbb{R}_t \times \mathbb{R}^6_x$*, and* γ^t *is the solution of classical dynamical system* (8.10)*.*

Remark 5 The closed 2-form $\tilde{\omega}_F = \omega_0 + F$, which appeared in (8.15), generates a contact structure on $\mathbb{R}^7 = \mathbb{R}_t \times \mathbb{R}^6_x$ [54, 55]. The "virtual" system (8.11) is the characteristic system for $\tilde{\omega}_F$. More precisely, the vector field on $\mathbb{R}^7$ corresponding to (8.11) is

$$v_0 = \frac{\partial}{\partial t} + eE(t,q)\frac{\partial}{\partial p}, \qquad x = (q,p).$$

This is the null-field for $\tilde{\omega}_F$:

$$v_0 \rfloor \tilde{\omega}_F = 0,$$

and the flow of v_0 preserves $\tilde{\omega}_F$:

$$\mathcal{L}_{v_0}\tilde{\omega}_F = 0.$$

Here we denote by $\mathcal{L}$ the Lie derivative and use the sign $\rfloor$ for the contraction of a vector field and a form: $v\rfloor\omega\,(u) \equiv \omega(u,v)$ for all u. If one denotes by v_H the vector field on $\mathbb{R}^7$ corresponding to (8.10)

$$v_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p}(q,p)\frac{\partial}{\partial q} + \Big(eE(t,q) - \frac{e}{c}B(t,q)\times\frac{\partial H}{\partial p}(q,p) - \frac{\partial H}{\partial q}(q,p)\Big)\frac{\partial}{\partial p},$$

then

$$v_H\rfloor\tilde{\omega}_F = dH - v_0(H)\,dt, \qquad \mathcal{L}_{v_H}\tilde{\omega}_F = d(v_0(H))\wedge dt.$$

Here $v_0(H) = eE\partial H/\partial p$; so we see how the electric field E determines the "nonconservation" properties of the charged particle dynamics in the contact space $\mathbb{R}^7 = \mathbb{R}_t\times\mathbb{R}^6_x$.

Now let us return to the WKB-representation (7.7), (7.11) of symbol $\mathcal{U}^t$ and calculate the Jacobian $\mathcal{J} = \mathcal{D}X/\mathcal{D}x^0$. The trajectory X of system (8.9) is given now by a modification of (7.12): $X(x^0,t) = \frac{1}{2}\big(\gamma^t(x^0)+\lambda^t(x^0)\big) + v^t(x^0)$, where v^t is the same as in (7.12). Since the solution $\lambda = \lambda^t$ of (8.11) is easily calculated: $\lambda^t_q = q^0$, $\lambda^t_p = p^0 + e\int_0^t E(\tilde{t},q^0)\,d\tilde{t}$, we derive

$$\mathcal{J} = \det\left[\frac{1}{2}(I+d\gamma^t) + \begin{pmatrix} 0 & 0 \\ C^t & D^t \end{pmatrix}\right]. \tag{8.16}$$

Here

$$C^t \equiv \frac{e}{c}\frac{\partial}{\partial q^0}\Big(\alpha^s\big(t,\gamma^t_q(x^0),q^0\big)\Big) + e\int_0^t \frac{\partial E(\tilde{t},q^0)}{\partial q^0}\,d\tilde{t}, \qquad D^t \equiv \frac{e}{c}\frac{\partial}{\partial p^0}\Big(\alpha^s\big(t,\gamma^t_q(x^0),q^0\big)\Big).$$

The function α^s is determined by $\alpha^s(t,q,q') \equiv \frac{1}{2}\big(\alpha(t,q,q')+\alpha(t,q',q)\big)$, where the two-point magnetic potential α is given by (8.4). The point x^0 everywhere in these formulas has to be expressed via t,x by means of the equation

$$x = \tfrac{1}{2}\big(\gamma^t(x^0)+\lambda^t(x^0)\big) + v^t(x^0). \tag{8.17}$$

This equation is uniquely solvable while the Jacobian is positive

$$\mathcal{J} \geq \delta > 0, \qquad t\in[0,T]. \tag{8.18}$$

So, we conclude with the following result.

Theorem 8 *The symbol $\mathcal{U}^t$ of the evolution operator* (8.5) *solving the equation of motion* (8.1a) *or* (8.1b) *can be represented (for sufficiently small time* (8.18)*) in the WKB-form* (7.7) *or* (8.9) *over the contact space $\mathbb{R}^7 = \mathbb{R}_t\times\mathbb{R}^6_x$. The phases S are given by membrane formula* (8.15) *and amplitudes $u^t = \mathcal{J}^{-1/2}$ by* (8.16).

Remark 6 Of course, the contact space $\mathbb{R}^7$ can be symplectified (see [54, 55]) up to $\mathbb{R}^8 = (\mathbb{R}_{p_0} \oplus \mathbb{R}_t) \times \mathbb{R}^6_x$ with symplectic form $\omega'_F = dp_0 \wedge dt + \tilde{\omega}_F$. The dynamical membrane $\Sigma_{t,x}$ in (8.15) can be blown up to a membrane $\Sigma'_{t,x}$ in such a way that the path γ^t is put on the level $p_0 = -H$, and the path λ^t is put on the level $p_0 = 0$; so, the summand $\int_0^t H\, d\tilde{t}$ in (8.15) is included into the membrane area, and altogether one obtains

$$S(t,x) = \int_{\Sigma'_{t,x}} \omega'_F.$$

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